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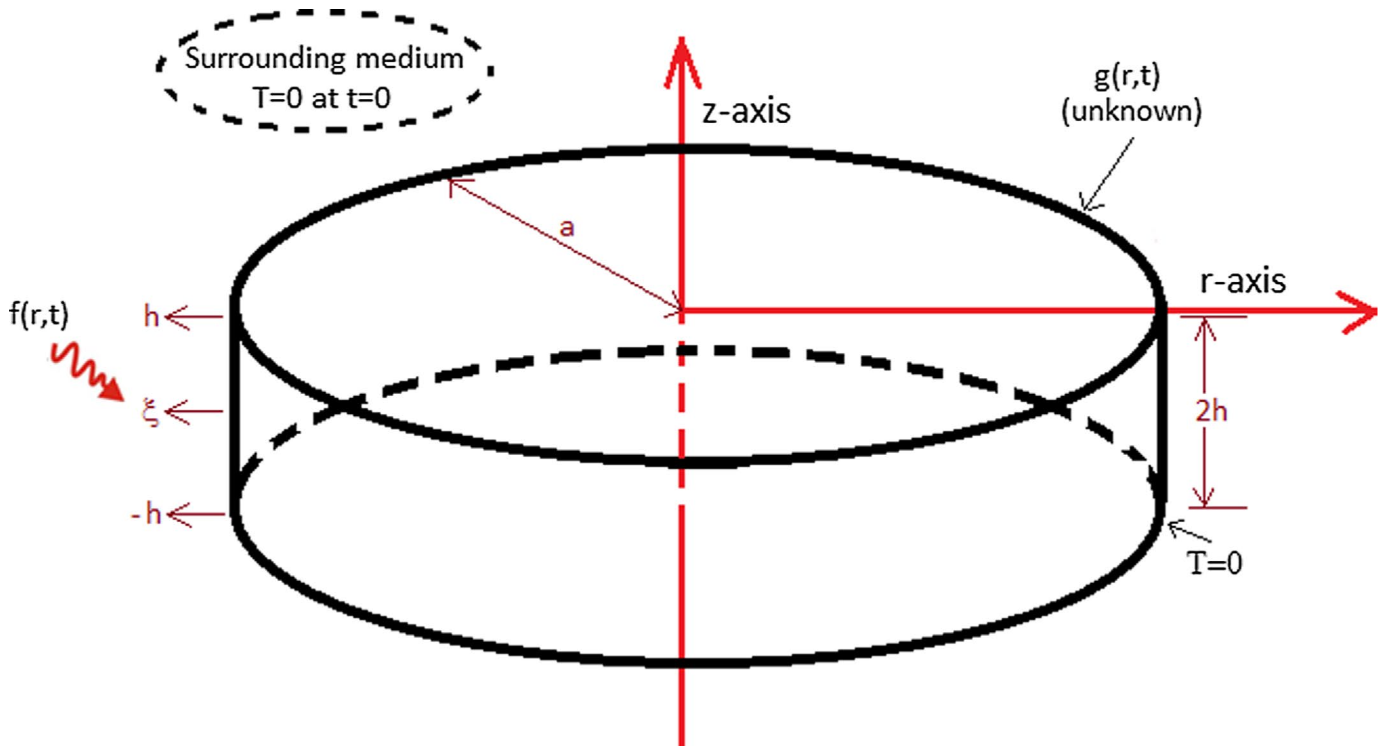

APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

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Inverse quasi–static unsteady–state thermal stresses in a thick circular plate

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Abstract: In this article, an attempt has been made to discuss the inverse thermoelastic problem of a thick circular plate defined as $0 \leq r \leq a$, $-h \leq z \leq h$ subjected to the arbitrary heat supply at interior point while circular edge of the thick circular plate at the outer surface and at the lower surface is maintained at zero temperature. The conductivity equation and the corresponding initial and boundary conditions have been solved using finite Hankel and Laplace integral transform techniques. Goodier's and Michell's functions are used to obtain the displacement components and its associated stresses. The results are obtained in a form in terms of Bessel's function. The results for unknown temperature, displacement, and stresses have been computed numerically considering special functions and illustrated graphically.

Subjects: Applied Mathematics; Applied Mechanics; Inverse Problems; Mathematical Modeling; Mathematics & Statistics; Science

Keywords: thick circular plate; thermoelasticity; unsteady-state; integral transform

AMS subject classifications: 35B07; 35G30; 35K05; 44A10

1. Introduction

As a result of the increased usage of industrial and construction materials, the interest in the inverse thermal stress problems has grown considerably, typified by main shaft of lathe and the role of the rolling mill due to the elementary geometry involved. As a result of this, a number of theoretical studies concerning them have been reported so far. However, to simplify this, almost all the studies were conducted on the assumption that the upper and lower surfaces of the circular plate are insulated or that the heat is dissipated with uniform heat transfer coefficients throughout the surfaces as direct problems. For example, Sabherwal (1965) investigated the inverse problem of transient

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Ishaque Khan is an assistant professor in Mathematics at M. G. College, Armori, Gadchiroli (MS), India and pursuing research work in the field of Thermoelasticity BVP. Lalsingh Khalsa has published 21 research papers and 5 books at international level, besides being an excellent supervisor. Vinod Varghese has also extended his full support in assisting the work and has lent a helping hand in providing it with a complete touch.

PUBLIC INTEREST STATEMENT

The results are useful to scientists working on two-dimensional thick plate theories, especially when circumstances are of inverse quasi-static transient conditions. Inverse problems are very crucial, especially in view of its relevance to various industrial machines and it can be the theoretical base for several geophysical and industrial applications. The physical applications are encountered in the context of problems such as ground explosions and oil industries, where the interest is in various phenomena occurring during the measurement of displacements, stresses, and temperature field due to the presence of certain sources.

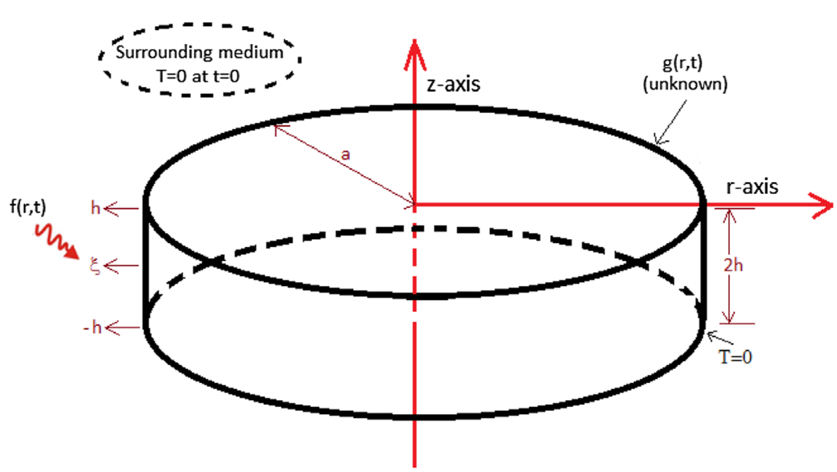
heat conduction in circular plate. Grysa, Ciałkowski, and Kamiński (1981) discussed on an inverse temperature filled problem of the theory of thermal stresses. Noda (1989) discussed an analytical method for an inverse problem of three-dimensional transient thermoelasticity in a transversely isotropic solid by integral transform technique with newly designed potential function and illustrated practical application of the method in engineering problem. Ashida, Choi, and Noda (1996) investigated an inverse thermoelastic problem in an isotropic structural plate onto which a piezoelectric ceramic plate is perfectly bonded. When unknown heating temperature acts on the free surface of the isotropic structural plate, an electric potential is induced in the piezoelectric ceramic plate. Deshmukh and Wanhede (1997, 1998a, 1998b) discussed the inverse transient problem of quasi-static thermal deflection in these clamped circular plates and axisymmetric inverse steady-state problem of thermoelastic deformation of finite length hollow cylinder and inverse quasi-static transient thermoelastic problem in a thin annular disk. Again Ashida et al. (2002) emphasized on the inverse transient thermoelastic problem for a composite circular disk. Yang, Chen, Chang (2002) studied inverse boundary value problem of coupled thermoelasticity in an infinitely long annular cylinder using simulated exact and inexact measurements. Patil and Prasad (2013) studied inverse steady-state thermoelastic problem of a thin rectangular plate using operational methods. From the previous literatures regarding thick plate as considered, it was observed by the author that no analytical procedure has been established for thick circular plate, considering inverse quasi-static thermoelastic analysis.

In this problem, we consider some new interesting results of the inverse heat conduction problem of thick circular plate occupying the space $D = \{(x, y, z) \in R^3 : 0 \le (x^2 + y^2)^{1/2} \le a, -h \le z \le h\}$, where $r = (x^2 + y^2)^{1/2}$. In a condition wherein a thick circular plate is subjected to arbitrary heat supply at interior point while the circular edge of the thick circular plate at the outer surface and at the lower surface is maintained at zero temperature, the governing heat conduction equation has been solved using integral transform method. The results are obtained in series form in terms of Bessel's functions. The mathematical model of final thick circular plate has been constructed with the help of numerical illustrations.

2. Formulation of the problem

Consider a thick circular plate of thickness 2 h occupying space D defined by $0 \le r \le a$, $-h \le z \le h$, as shown in Figure 1. Let the plate be subjected to an arbitrary known interior temperature $f(r, t)$ within the region $-h \le z \le h$. With lower surface and circular surface, $r = a$ at zero temperature. Under this more realistic prescribed condition, the unknown temperature $g(r, t)$ which is at the upper surface of the plate $z = h$ and quasi-static thermal stresses due to unknown temperature $g(r, t)$ are to be determined.

Figure 1. Geometrical configuration of the problem.



2.1. Temperature distribution

The transient heat conduction equation of the plate is given as follows:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

where κ is the thermal diffusivity of the material of the disk (which is assumed to be constant), subjected to the initial and boundary conditions:

$$T = 0 \quad \text{at} \quad t = 0 \tag{2}$$

$$T = 0 \quad \text{at} \quad r = a \tag{3}$$

$$T = 0 \quad \text{at} \quad z = -h \tag{4}$$

$$T = f(r,t) \quad \text{at} \quad z = \xi \tag{5}$$

$$T = g(r,t) \quad \text{at} \quad z = h \quad 0 \le r \le a \qquad (\text{Unknown}) \tag{6}$$

2.2. Thermal displacements and thermal stresses

Following Noda et al. (2003), we assume that the Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem as:

$$\begin{aligned} &\nabla^2 U_r - \frac{U_r}{r} + \frac{1}{1-2\upsilon}\frac{\partial e}{\partial r} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial T}{\partial r} = 0 \\ &\nabla^2 U_z - \frac{1}{1-2\upsilon}\frac{\partial e}{\partial z} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial T}{\partial z} = 0 \end{aligned} \tag{7}$$

where U_r and U_z are the displacement components in the radial and axial directions, respectively, and the dilatation e as:

$$e = \frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z} \tag{8}$$

The displacement function in the cylindrical coordinate system is represented by Goodier's thermoelastic displacement potential ϕ and Michel's function M:

$$U_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z}, \tag{9}$$

$$U_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 M - \frac{\partial^2 M}{\partial^2 z} \tag{10}$$

in which Goodier's thermoelastic potential must satisfy:

$$\nabla^2 \phi = K\tau \text{ with } \phi = 0 \text{ at } t = 0. \tag{11}$$

and the Michel's function M must satisfy:

$$\nabla^2 \nabla^2 M = 0 \tag{12}$$

in which K is the restraint coefficient and temperature change $\tau = T - T_{i,}$ T_i is the initial temperature, and:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The component of the stresses is represented as:

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \varphi}{\partial r^2} - k\tau + \frac{\partial}{\partial z}\left(\nu\,\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{13}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \varphi}{\partial r} - k\tau + \frac{\partial}{\partial z}\left(\nu\,\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{14}$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \phi}{\partial z^2} - k\tau + \frac{\partial}{\partial z}\left((2-\mathrm{v})\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{15}$$

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\mathrm{v})\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{16}$$

where G and ν are shear modulus and Poisson's ratio, respectively; for the traction-free surface, the stress function is:

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } r = a \tag{17}$$

Equations (1) to (17) constitute the mathematical formulation of the problem.

3. Solution of the problem

3.1. Solution for temperature distribution

In order to solve Equation (1) under the boundary condition (3), we firstly introduce the finite Hankel transform of order m over the variable r; the integral transform and its inversion theorem (Sneddon, 1972) can be written as:

$$\left.\begin{aligned} \bar{T}(\alpha_m, z, t) &= \int_0^a rJ_0(\alpha_m, r)T(r,z,t)dr, \\ T(r,z,t) &= \sum_{m=1}^{\infty}\left(\frac{2J_0(\alpha_m, r)}{a^2 J_1^2(\alpha_m a)}\right)\bar{T}(\alpha_m, z, t) \end{aligned}\right\} \tag{18}$$

in which α_1, α_2, α_3, ... are the roots of the transcendental equation $J_0(\alpha_m a) = 0$.

Applying the finite Hankel transform and Laplace integral transform, and its inversion theorems, results in the final temperature distribution as:

$$T(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4(-1)^{n+1} n\pi\, k\, J_0(\alpha_m r)\,\sin(\beta_n)}{a^2(\xi+h)^2 J_1^2(\alpha_m a)} \int_0^t \exp(-k\,\wp_{n,m}\,u)\,\bar{f}(\alpha_m,\, t-u)\,du \tag{19}$$

The function given in Equation (23) represents the temperature at every instant and at all points of the circular thick plate of finite height.

The unknown temperature $g(r, t)$ can be obtained by substituting $z = h$ in the equation.

$g(r, t) = T\,(r\ z, t)$ at $z = h$, one obtains:

$$g(r,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} n\pi\, k\, J_0(\alpha_m r)\,\sin(\beta_n)}{a^2(\xi+h)^2 J_1^2(\alpha_m a)} \int_0^t \exp(-k\,\wp_{n,m}\,u)\,\bar{f}(\alpha_m,\, t-u)\,du \tag{20}$$

in which:

$$\beta_n = n\pi\left(\frac{z+h}{\xi+h}\right),\ \wp_{n,m} = \alpha_m^2 + \left(\frac{n\pi}{\xi+h}\right)^2$$

3.2. Solution for thermal stresses

3.2.1. Goodier thermoelastic displacement potential (ϕ)

Referring to the fundamental Equation (1) and its solution (19) for the heat conduction problem, the solution for the displacement function is represented by Goodier's thermoelastic displacement potential ϕ governed by Equation (11); this is represented by:

$$\varphi(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4kJ_0(\alpha_m r)}{a^2J_1^2(\alpha_m a)}\frac{(-1)^n\sin(\beta_n)}{(\xi+h)^2\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m})u\,\bar{f}(\alpha_m,\, t-u)]\,du \tag{21}$$

3.2.2. Michell's function M

Similarly, the solution for Michell's function M is assumed so as to satisfy the governed condition of Equation (12) as:

$$M = \frac{4k}{a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_0(\alpha_m r)}{J_0^2(\alpha_m a)}\{H_{mn}\ \sinh[\alpha_m(z+h)] + R_{mn}\,\alpha_m(z+h)\cosh[\alpha_m(z+h)]\} \tag{22}$$

in which H_{mn} and R_{mn} are arbitrary functions.

3.2.3. Displacement and thermal stresses

In this manner, two displacement functions in the cylindrical coordinate system ϕ and M are fully formulated. Now in order to obtain the displacement components, we substitute the values of thermoelastic displacement potential φ and Michell's function M in Equations (9) and (10), which results in:

$$\begin{aligned}U_r = \left(\frac{4k}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-\alpha_m)(-1)^n n\pi\,J_1(\alpha_m r)\sin(\beta_n)}{a^2(\xi+h)^2\wp_{n,m}J_1^2(\alpha_m a)}\left\{\int_0^t[\exp(-k\wp_{n,m}\,u)\ \bar{f}(\alpha_m,\, t-u)]\,du\right.\\ \left.+H_{mn}\,\alpha_m^2\ \cosh[\alpha_m(z+h)] + R_{mn}\,\alpha_m^2\left\langle\cosh[\alpha_m(z+h)] + \alpha_m(z+h)\ \sinh[\alpha_m(z+h)]\right\rangle\right\}\end{aligned} \tag{23}$$

$$\begin{aligned}U_z = \left(\frac{4k^2}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n n^2\pi^2 J_0(\alpha_m r)}{(\xi+h)^2a^2J_1^2(\alpha_m a)}\left\{\cos(\beta_n) - H_{mn}\,\alpha_m^2\,\sinh[\alpha_m(z+h)]\right.\\ \left.+R_{mn}\alpha_m^2\,\langle 2\,(1-2v)\,\sinh[\alpha_m(z+h)] - \alpha_m(z+h)\cosh h[\alpha_m+h]\rangle\right\}\end{aligned} \tag{24}$$

$$\begin{aligned}\sigma_{rr} = \left(\frac{4kG}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\alpha_m a)}\Bigg\{\quad & 2(\alpha_m J_0(\alpha_m r) - r^{-1}J_1(\alpha_m r))\\ &\times\frac{(-1)^{n+1}n\pi\sin(\beta_n)}{(\xi+h)^2\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m}\,u)\ \bar{f}(\alpha_m,\, t-u)]\,du\\ &-k\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4k\,(-1)^{n+1}n\pi\,J_0(\alpha_m r)\sin(\beta_n)}{a^2\,J_1^2(\alpha_m a)\,(\xi+h)^2}\int_0^t[\exp(-k\wp_{n,m}\,u)\ \bar{f}(\alpha_m,\, t-u)]\,du\\ &+H_{mn}\,\alpha_m^2(\alpha_m J_0(\alpha_m r) - r^{-1}J_1(\alpha_m r))\cosh[\alpha_m(z+h)]\\ &+R_{mn}\,\alpha_m^2\,[2\upsilon\,\alpha_m, J_0(\alpha_m r)\cosh[\,\alpha_m(z+h)] + (\alpha_m J_0(\alpha_m r) - r^{-1}J_1(\alpha_m r))\\ &\times\langle\cosh[2\alpha_m(z+h)]\sinh[\alpha_m(z+h)]\,\rangle\Bigg\}\end{aligned} \tag{25}$$

$$\sigma_{\theta\theta}=\left(\frac{4k}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_1(\alpha_m,r)}{J_1^2(\alpha_m a)}\left\{\frac{2(-1)^{n+1}n\pi\alpha_m\sin(\beta_n)F(t)}{(\xi+h)^2\wp_{n,m}}\right.$$
$$-k\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4k(-1)^{n+1}n\pi J_0(\alpha_m r)\sin(\beta_n)}{a^2J_1^2(\alpha_m a)(\xi+h)}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du$$
$$+H_{mn}r^{-1}\alpha_m^2J_1(\alpha_m r)\cosh[\alpha_m(z+h)]+R_{mn}\alpha_m^2\langle 2v\,\alpha_m J_0(\alpha_m r)$$
$$\left.\times\cosh[\alpha_m(z+h)]+r^{-1}J_1(\alpha_m r)(\cosh[\alpha_m(z+h)]+\alpha_m(z+h)\sinh[\alpha_m(z+h)])\rangle\right\} \tag{26}$$

$$\sigma_{zz}=\left(\frac{4k}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}n^2\pi^3J_0(\alpha_m r)\sin(\beta_n)}{(\xi+h)^4J_1^2(\alpha_m a)\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du$$
$$-k\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^{n+1}n\pi J_0(\alpha_m r)}{a_2(\xi+h)^2J_1^2(\alpha_m a)}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du \tag{27}$$
$$-H_{mn}\alpha_m^3\cosh[\alpha_m(z+h)]+R_{mn}\alpha_m^3\langle(1-2v)\cosh[\alpha_m(z+h)]$$
$$-\alpha_m(z+h)\sinh[\alpha_m(z+h)]\rangle\}$$

$$\sigma_{rz}=\left(\frac{4k}{a^2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_1(\alpha_m r)}{J_0^2(\alpha_m a)}\left\{\frac{(-1)^{n+1}2n^2\pi^2\cos(\beta_n)}{(\xi+h)^3\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du\right.$$
$$+H_{mn}\alpha_m^3\sinh[\alpha_m(z+h)]+R_{mn}\alpha_m^3\langle 2v\sinh[\alpha_m(z+h)]$$
$$\left.+\alpha_m(z+h)\cosh[\alpha_m[\alpha_m(z+h)]]\rangle\right\} \tag{28}$$

3.2.4. Determination of unknown arbitrary function H_{mn} and R_{mn}

Applying boundary condition Equation (17) to Equations (25) and (28), one obtains:

$$R_{mn}=\frac{(-1)^n2n^2\pi^2\cos(\beta_n)}{(\xi+h)^3\alpha_m^3\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du$$
$$\times\frac{1}{\sinh[\alpha_m(z+h)]+\alpha_m(z+h)\cosh[\alpha_m(z+h)]} \tag{29}$$

$$H_{mn}=\frac{(-1)^{n+1}2n^2\pi^2\cos(\beta_n)}{(\xi+h)^3\alpha_m^3\wp_{n,m}}\int_0^t[\exp(-k\wp_{n,m}u)\,\bar{f}(\alpha_m,t-u)]\,du$$
$$\times\frac{1-2\upsilon}{\sinh[\alpha_m(z+h)]+\alpha_m(z+h)\cosh[\alpha_m(z+h)]} \tag{30}$$

4. Special case and numerical calculations

$$\text{Setting } f(r,t)=(r^2-a^2)^2(1-e^{-t}) \tag{31}$$

Applying finite Hankel transform to the Equation (31), one attains:

$$\bar{f}(\alpha_m,t)=\int_0^a r(r^2-a^2)^2J_0(\alpha_m,r)(1-e^{-t})dr$$
$$=8a\,\{(8-a^2\alpha_m^2)J_1(\alpha_m a)-4a\alpha_mJ_0(\alpha_m a)\}\,(1-e^{-t})/\alpha_m^5 \tag{32}$$

Figure 2. Temperature distribution.

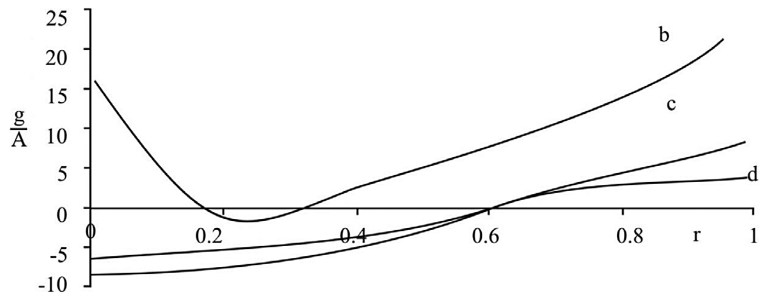


Figure 3. Radial displacement profile.

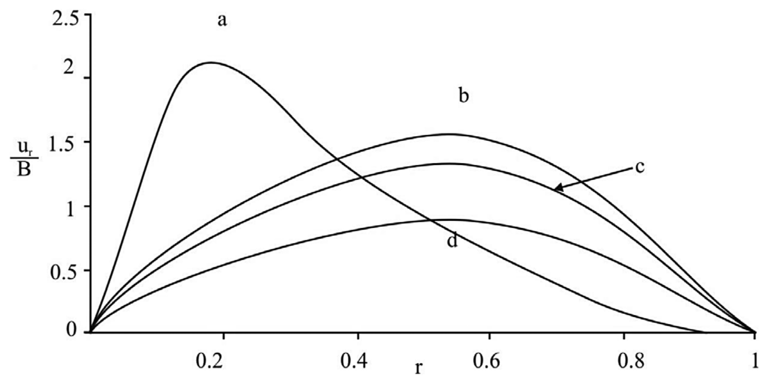


Figure 4. Axial displacement profile.

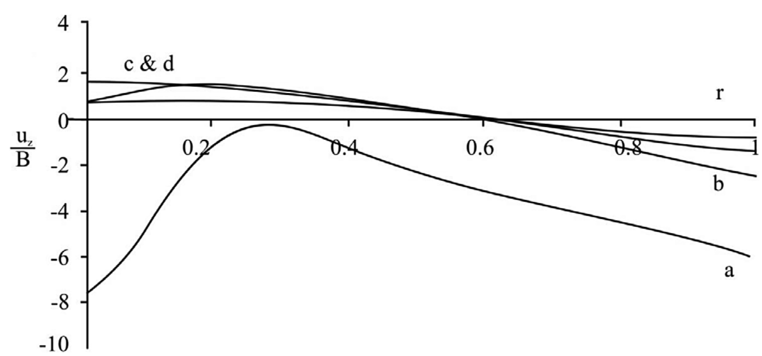
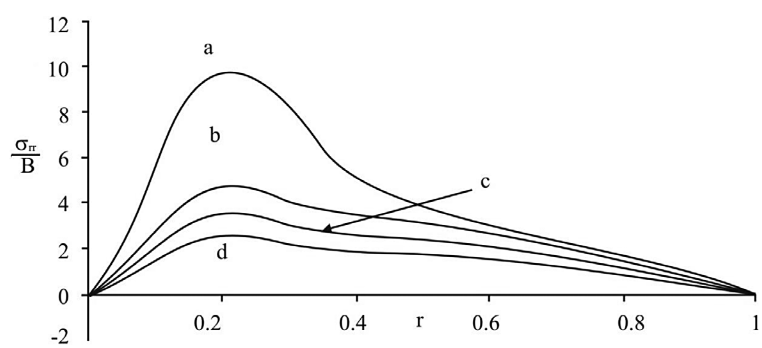


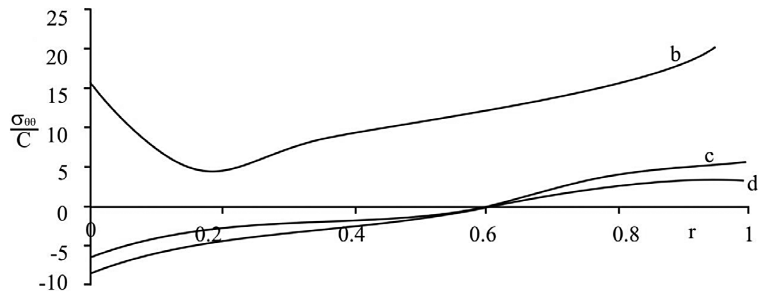
Figure 5. Radial stress distribution.



5. Numerical calculations

The numerical calculation has been carried out for steel (SN 50C) plate with parameters $a = 1\ m$, $h = 0.2\ m$, thermal diffusivity $k = 15.9 \times 10{-}6$ (m^2s^{-1}), and Poisson's ration $v = 0.281$, with $\alpha_1 = 3.8317$,

Figure 6. Tangential stress distribution.



$\alpha_2 = 7.0156$, $\alpha_3 = 10.1735$, $\alpha_4 = 13.3237$, $\alpha_5 = 16.470$, $\alpha_6 = 19.6159$, $\alpha_7 = 22.7601$, $\alpha_8 = 25.9037$, $\alpha_9 = 29.0468$, and $\alpha_{10} = 32.18$ being the roots of transcendental equation $J_0(\alpha_m a) = 0$.

For convenience, set $A = -16/10^2 a$, $B = 16K/10^2 a$, and $C = 32GK/10^2 a$ in the expressions for obtaining the unknown temperature, displacement, and stress components.

In order to examine the influence of unknown temperature on the upper surface of circular plate, the numerical calculation $z = h/2$, $r = 0$, 0.2, 4.0, 6.0, 8.0, & 1 and $\xi = -0.2$, -0.1 and 0.1 was performed. Numerical variations in radial directions have been illustrated in the figure with the help of a computer program.

6. Concluding remarks

In this problem, a thick circular plate is considered which is kept traction-free as well as subjected to arbitrary known interior temperature and determined for the expressions of unknown temperature, displacements, and stress functions due to the unknown temperature. As a special case, mathematical model is constructed for $f(r) = (r^2 - a^2)^2 (1 - e^t)$ and numerical calculations were performed. The thermoelastic behaviors such as temperature, displacements, and stresses are examined with the help of arbitrary known interior temperature along the radial direction as $a \to -0.2$, $b \to -0.1$, $c \to 0$, $d \to 0.1$.

Figure 2 indicates that the unknown temperature decreases from $r = 0$ to $r = 0.3$ and increases from 0.3 to 1 with the thickness of the circular plate. As the source of known temperature varies from a negative to positive value, the unknown temperature decreases its magnitude along the radial direction.

As shown in Figure 3, the source of known temperature varies from bottom to top, the radial displacement decreases at $r = 0$, and the radial displacement vanishes; otherwise, its existence would have been visible.

As shown in Figure 4, the source of known temperature varies from bottom to top; the axial displacement increases along radial direction; and it shows its existence.

Figure 5 shows that the radial stress decreasing from bottom to (lower surface to upper surface) top. Stress at $r = 0$ and $r = a$ is zero; otherwise, it shows its existence.

Figure 6 indicates that the stress function $\sigma_{\theta\theta}$ decreases with the thickness of the circular plate. It shows the existence for small thickness. Also, it develops tensile stresses in the radial direction.

In this article, we analyzed an inverse thermoelastic problem of a thick circular plate and determined the expressions of unknown temperature, displacement, and thermal stresses. The heat conduction differential equation is solved using finite Hankel and Laplace integral transform techniques, and their inversion theorems. Goodier's and Michell's functions are used to obtain the displacement components. As a special case, a mathematical model is constructed for steel (SN 50C) thick plate, with the material properties specified as above, and examined for the thermoelastic behaviors in unsteady-state field for unknown temperature change, displacement, and thermal stresses. We conclude that the displacement and stress components occur near the heat source region. As the temperature increases, the circular plate will tend to expand in the radial direction as well as in the axial direction. Also any particular case of special interest can be derived by assigning values to the parameters and functions in the expressions (19)–(28).

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Research Article

Quasi-Static Transient Thermal Stresses in an Elliptical Plate due to Sectional Heat Supply on the Curved Surfaces over the Upper Face

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Abstract. This paper is an attempt to determine quasi-static thermal stresses in a thin elliptical plate which is subjected to transient temperature on the top face with zero temperature on the lower face and the homogeneous boundary condition of the third kind on the fixed elliptical curved surface. The solution to conductivity equation is elucidated by employing a classical method. The solution of stress components is achieved by using Goodier's and Airy's potential function involving the Mathieu and modified functions and their derivatives. The obtained numerical results are accurate enough for practical purposes, better understanding of the underlying elliptic object, and better estimates of the thermal effect on the thermoelastic problem. The conclusions emphasize the importance of better understanding of the underlying elliptic structure, improved understanding of its relationship to circular object profile, and better estimates of the thermal effect on the thermoelastic problem.

Keywords: Elliptical plate, Temperature distribution, Thermal stresses, Mathieu function.

1. Introduction

The theoretical study of the heat flow within hollow elliptical structures has been of considerable practical importance in a wide range of sectors such as mechanical, aerospace, and food engineering fields for the past few decades. Unfortunately, there are only a few studies concerned with steady and transient state heat conduction problems in the elliptical objects. A short history of the research work associated with the thermoelasticity provides an insight into various approximate methods like the Ritz energy method, Galerkin's Method, finite element models, and the perturbation theory to solve the system. In the most recent literature, some researchers have undertaken studies on heat conduction analysis, which can be summarized as given below. Gupta [1] introduced a finite transform involving the Mathieu functions used for obtaining the solutions for boundary value problems involving elliptic cylinders. Sato [2] subsequently obtained the mathematical solution for the heat conduction problem of an infinite elliptical cylinder during heating and cooling considering the effect of the surface resistance. Recently, El Dhaba [3] used a boundary integral method to solve the problem of the plane uncoupled linear thermoelasticity with heat sources for an infinite cylinder with elliptical cross section which was subjected to a uniform pressure and a thermal radiation condition on its boundary. However, a few studies have been done to eliminate the thermoelastic problems successfully. Most recently, Helsing [4] formulated an elastic problem with mixed boundary conditions, that is, Dirichlet conditions on parts of

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the boundary and Neumann conditions on the remaining contiguous parts, and solved it on an interior planar domain using an integral equation method. Dang and Mai [5] estimated mixed boundary value problem for a biharmonic equation of the Airy stress function which modeled a crack problem of a solid elastic plate using an iterative method. Al Duhaim et al. [6] determined the thermal stress of a mixed boundary value problem in half space by using Jones's modification of the so-called Wiener-Hopf technique. Parnell et al. [7] employed the Wiener-Hopf and Cagniard-de Hoop techniques to solve a range of transient thermal mixed boundary value problems on the half-space. Nuruddeen and Zaman [8] obtained the analytical solution of transient heat conduction in a solid homogeneous infinite circular cylinder using the Wiener-Hopf technique owing to the mixed nature of the boundary conditions. Very recently, Bhad [9,10] has obtained a few thermoelastic solutions for elliptical objects using the integral transform technique. The above reviews clearly suggest that, in contrast with the classical circular or rectangular structures case, nearly all investigators so far have focused on the thermoelastic problems in elliptical membranes either in the steady or unsteady state. In particular, there seems to be no rigorous analytical or numerical reports on the quasi-static transient response of a thin elliptical plate subjected to a thermal load. The primary purpose of the current work is to fill this gap. Both analytical and numerical techniques can be the best methodology to solve such problems. Nevertheless, it is observed that mostly numerical solutions are preferred due to either non-availability or mathematical complexity of the corresponding exact solutions. Rather, limited utilisation of analytical solutions should not diminish their merit over numerical ones; since exact solutions, if available, provide an insight into the governing physics of the problem that is often missing in any numerical solution. However, to the best of authors' knowledge, very few works have been published to determine the temperature distribution and its associated stresses in an elliptical plate with boundary conditions of radiation type on the outside surfaces with independent radiation constants. Owing to the lack of research in elliptical objects in the elliptic-cylinder coordinate system, the authors have been motivated to conduct this study using the classical method.

The object of this paper is to study the quasi-static thermal stresses in a thin elliptical plate subjected to a sectional heat supply on the upper face with the lower face kept at zero temperature. To establish the quasi-static problem formulation, the following assumptions need to be made: (i) The material of the cylinder is elastic, homogeneous, and isotropic, (ii) A thin-walled cylinder has been considered during the investigation with a ratio of the length to the thickness greater than 8, (iii) The deflection (the normal component of the displacement vector) of the mid-plane is small as compared with the thickness of the plate, and (iv) The stress perpendicular to the middle plane is small compared with the other stress components and may be neglected in the stress-strain relations. The success of this research mainly lies in the analytical procedures which present a much simpler approach for optimisation of the design regarding the material usage and performance in the engineering problem, particularly in the determination of the thermoelastic behaviour in the elliptical disc engaged as the foundation of pressure vessels, furnaces, etc. Actually, by considering a circle as a special kind of ellipse, it is shown that the temperature distribution and history in a circular solution can be derived as a special case of the present mathematical solution for the elliptical disc.

2. Formulation of the problem

It is assumed that a thin elliptical plate is occupying the space $D:\{(\xi,\eta,z)\in R^3:\ 0<\xi<\xi_0,\ 0<\eta<2\pi,\ -\ell/2<z<\ell/2\}$ under unsteady-state temperature field with no internal heat source within it. The geometry of the plate as shown in Fig. 1 indicates that an elliptic coordinate system (ξ,η,z) is the most appropriate choice of the reference frame, which is related to the rectangular coordinate system (x,y,z) by the relation $x=c\cosh\xi\cos\eta$, $y=c\sinh\xi\sin\eta$, $z=z$.

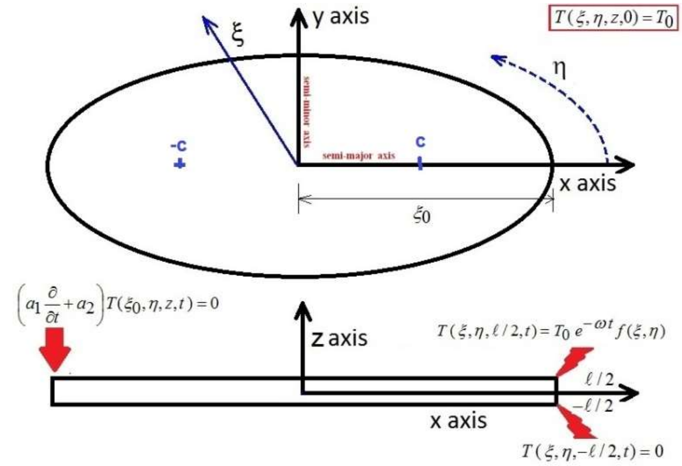

Fig. 1. Physical configuration of the elliptical plate

The length $2c$ is the distance between its common foci as shown in the geometrical configuration described in Fig. 1 which can be defined as $2c=2\sqrt{a^2-b^2}$. The curves $\eta=$ constant represent a family of confocal hyperbolas, while the curves

$\xi =$constant constitute a family of confocal ellipses. Both sets of curves intersect each other orthogonally at every point in space. The geometry parameters are given as $\xi \in [0,\xi_0]$, $\eta \in [0,2\pi)$, and $z \in [-\ell/2,\ell/2]$. The plate is subjected to the arbitrary initial temperature over the upper surface $(z=\ell/2)$ with the lower surface $(z=-\ell/2)$ at zero temperature and boundary condition of the third kind on the curved surface; Under these prescribed conditions, the quasi-static thermal stresses are required to be determined.

2.1 Heat conduction of the problem

The governing differential equation for heat conduction and boundary conditions can be defined as:

$$\frac{1}{\kappa}\frac{\partial}{\partial t}(\xi,\eta,z,t) = h^2\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)T(\xi,\eta,z,t)+\frac{\partial^2}{\partial z^2}T(\xi,\eta,z,t) \tag{1}$$

$$\left(a_1\frac{\partial}{\partial t}+a_2\right)T(\xi_0,\eta,z,t)=0 \tag{2}$$

$$T(\xi,\eta,\ell/2,t)=T_0\, e^{-\omega t} f(\xi,\eta) \tag{3}$$

$$T(\xi,\eta,-\ell/2,t)=0 \tag{4}$$

in which $T(\xi,\eta,z,t)$ is the temperature of the plate at point (ξ,η,z) at t time, T_0 is the temperature at time $t=0$ on the circumference of the elliptical plate of radius ξ_0 on the upper face, $a_i\,(i=1,2)$ are radiation coeffiecnts, $\omega > 0$ is a constant, λ is the coefficient of thermal conductivity, $\kappa = \lambda/\rho C$ represents thermal diffusivity in which λ is the thermal conductivity of the material, ρ is the density, C is the calorific capacity which is assumed to be constant, and h is the metric coefficient given by:

$$h^2 = 2/[c^2(\cosh 2\xi - \cos 2\eta)] \tag{5}$$

and the ∇^2 denotes the two-dimensional Laplacian operator

$$\nabla^2 = h^2\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right) \tag{6}$$

2.2 Associated thermal stress problem

The medium is defined by $0 \le \xi \le \xi_0$, $0 \le \eta \le 2\pi$, and $-\ell/2 \le z \le \ell/2$, and compiling various boundary conditions in the elliptical coordinates are defined to determine the influence of thermal boundary conditions on the thermal stresses. Since it is assumed that the cylinder is sufficiently thin, we can introduce the assumption that the plane, initially normal to the middle or neutral plane $(z=0)$ before bending, remains straight and normal to the middle surface during the deformation, and the length of such elements are not altered. This means that the axial stress, which is considered negligible compared with the other stress components, may be neglected in the stress-strain relations. Thus, for solving the quasi-static thermoelasticity problem by using the displacement potential method [11], we assume the potential function $\phi(\xi,\eta,z,t)$ such that it satisfies the equation given below:

$$h^2\nabla^2\phi = \frac{1+\upsilon}{1-\upsilon}\alpha_t T \tag{7}$$

in which G is the shear modulus, υ denotes Poisson's ratio, and α_t is the coefficient of linear expansion, respectively.

The components of the stresses are represented by the use of Goodier's potential stress function $\phi(\xi,\eta,z,t)$ as:

$$\left.\begin{aligned}
(1/h^4)\bar{\sigma}_{\xi\xi} &= -2G(c^2/2)[(\cosh 2\xi - \cos 2\eta)\phi_{,\eta\eta} + \sinh 2\xi\ \phi_{,\xi} - \sin 2\eta\ \phi_{,\eta}],\\
(1/h^4)\bar{\sigma}_{\eta\eta} &= -2G(c^2/2)[(\cosh 2\xi - \cos 2\eta)\phi_{,\xi\xi} - \sinh 2\xi\ \phi_{,\xi} + \sin 2\eta\ \phi_{,\eta}],\\
(1/h^4)\bar{\sigma}_{\xi\eta} &= -2G(c^2/2)[-(\cosh 2\xi - \cos 2\eta)\phi_{,\xi\eta} + \sin 2\xi\ \phi_{,\eta} + \sinh 2\eta\ \phi_{,\xi}]
\end{aligned}\right\} \tag{8}$$

It is observed that the displacements and stresses obtained from equations (7) and (8) do not satisfy the boundary conditions, i.e., the plate should be stress-free. To complement the solution, we found out that the complementary stresses $\bar{\bar{\sigma}}_{ij}$ satisfying the following relations:

$$\bar{\sigma}_{\xi\xi} + \bar{\bar{\sigma}}_{\xi\xi} = 0, \quad \bar{\sigma}_{\xi\eta} + \bar{\bar{\sigma}}_{\xi\eta} = 0 \quad \text{on} \quad \xi = a \tag{9}$$

To solve the isothermal elasticity problem, let us make use of the Airy potential stress function $\chi(\xi,\eta,z,t)$ which satisfies the bi-laplacian equation as:

$$[h^2\nabla^2\chi)]^2 = 0 \tag{10}$$

Then, the complementary stresses in terms of the Airy stress function are given by

$$\begin{aligned}
(1/h^4)\bar{\bar{\sigma}}_{\xi\xi} &= (c^2/2)[(\cosh 2\xi - \cos 2\eta)\,\chi,_{\eta\eta} + \sinh 2\xi\ \chi,_{\xi} - \sin 2\eta\ \chi,_{\eta}], \\
(1/h^4)\bar{\bar{\sigma}}_{\eta\eta} &= (c^2/2)[(\cosh 2\xi - \cos 2\eta)\chi,_{\xi\xi} - \sinh 2\xi\ \chi,_{\xi} + \sin 2\eta\ \chi,_{\eta}], \\
(1/h^4)\bar{\bar{\sigma}}_{\xi\eta} &= (c^2/2)[-(\cosh 2\xi - \cos 2\eta)\chi,_{\xi\eta} + \sin 2\xi\ \chi,_{\eta} + \sinh 2\eta\ \chi,_{\xi}]
\end{aligned} \tag{11}$$

Thus, the final stresses can be represented as

$$\begin{aligned}
\sigma_{\xi\xi} &= \bar{\sigma}_{\xi\xi} + \bar{\bar{\sigma}}_{\xi\xi} \\
\sigma_{\eta\eta} &= \bar{\sigma}_{\eta\eta} + \bar{\bar{\sigma}}_{\eta\eta} \\
\sigma_{\xi\eta} &= \bar{\sigma}_{\xi\eta} + \bar{\bar{\sigma}}_{\xi\eta}
\end{aligned} \tag{12}$$

The equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

3. Solution to the problem

3.1 Solution to the temperature field

We assume that the temperature distribution $T(\xi,\eta,z,t)$ is given by:

$$T = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} f_n(t) \sin\left[q_{n,m}\left(z + \frac{\ell}{2}\right) + \left(z + \frac{\ell}{2}\right)\right] Ce_{2n}(\xi, q_{n,m}) ce_{2n}(\eta, q_{n,m}) \tag{13}$$

in which the function $f_n(t)$ for all time t will be determined later on from the heat conduction equation, $ce_n(\eta,q)$ is the ordinary Mathieu function of the first kind of order n, and $Ce_n(\xi,q)$ is the modified Mathieu function of the second kind of order n. In this case, it is found that the temperature on the face $z = -h/2$ is zero.

By using equation (13) in equation (1), one obtains:

$$f_n(t),_t = -\kappa\alpha_{n,m}^2 f_n(t) \tag{14}$$

in which

$$\alpha_{n,m}^2 = 4q_{n,m}/c^2$$

On integrating equation (14), one obtains

$$f_n(t) = A_{n,m} \exp(-\kappa\alpha_{n,m}^2 t) \tag{15}$$

in which the constant $A_{n,m}$ can be determined from the nature of temperature prescribed on the upper face. Now, by using condition (3), one obtains the function $f(\xi,\eta)$ by the well-known theorem on Fourier's series [1, pp. 296] which can be expressed as:

$$\zeta(\xi,\eta) = T(\xi,\eta,\ell/2,0) = \ell \sum_{n=0}^{\infty}\left[\sum_{m=1}^{\infty} A_{2n,m} Ce_{2n}(\xi, q_{2n,m}) ce_n(\eta, q_{2n,m})\right] \tag{16}$$

Both sides of equation (16) are multiplied by $(\cosh 2\xi - \cos 2\eta)\ Ce_p(\xi, q_{p,r})\ ce_p(\eta, q_{p,r})$ and integrated with respect to η from 0 to 2π, and with respect to ξ from 0 to a. Then, by using orthogonal property [1, pp. 175-176], all terms vanish except when $p = n,\ r = m$. Hence,

$$A_{2n,m} = \frac{T_0 \exp(-\omega t)\int_0^a\int_0^{2\pi} f(\xi) Ce_{2n}(\xi, q_{2n,m}) ce_n(\eta, q_{2n,m})[\cosh 2\xi - \cos 2\eta]\, d\xi\, d\eta}{\pi\ell\int_0^a (\cosh 2\xi - \Theta_{2n,m}) Ce_{2n}^2(\xi, q_{2n,m})\, d\xi} \tag{17}$$

and

$$\Theta_{2n,m} = \frac{1}{\pi}\int_0^{2\pi} \cos 2\eta \, ce_{2n}^2(\eta, q_{2n,m})\, d\eta$$
$$= A_0^{(2n)} A_2^{(2n)} + \sum_{r=0}^{\infty} A_{2r}^{(2n)} A_{2r+2}^{(2n)}$$

in which

$ce_{2n}(\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\eta$ is a Mathieu function [1, pp. 27],

$Ce_{2n}(\xi, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2r\xi$ is a modified Mathieu function [1, pp. 21],

$q_{2n,m}$ is the root of the transcendental equation $Ce_{2n}(b,q)Fey_{2n}(a,q) - Fey_{2n}(b,q)Ce_{2n}(a,q) = 0$,

and the recurrence relations for the Bessel functions Y_{2r} can be defined as

$$Fey_{2n}(\xi, q) = \sum_{r=0}^{\infty} (A_{2r}^{(2n)} / A_0^{(2n)})\, ce_{2n}(0,q) Y_{2r}(2k \sinh \xi)\ [|\sinh \xi| > 1; R(\xi) > 0]\,.$$

The function given in equation (1) represents the temperature at every instant and all points of a thin elliptical plate of finite height which is subjected to transient temperature on the upper surface $(z = \ell/2)$ with the lower surface $(z = -\ell/2)$ at zero temperature, and the curved surface has homogeneous boundary conditions of the third kind.

3.2 Solution to the thermal stresses

By referring to the fundamental equation (1) and its solution (13) for the heat conduction equation, the solution for the displacement function is represented by Goodier's displacement potential $\phi(\xi, \eta, z, t)$, referred to by equation (8) as

$$\phi = -\alpha t\left(\frac{1+\nu}{1-\nu}\right)c^2h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \overline{f}(q_{n,m}) \sin\left[q_{n,m}\left(z+\frac{h}{2}\right)+\left(z+\frac{h}{2}\right)\right]$$
$$\times C_{e_{2n}}(\xi, q_{n,m}) c_{e_{2n}}(\eta, q_{n,m}) \exp[(-k\alpha_{2n,m}^2 + \omega)t] / 4q_{2n,m} C_{n,m} \tag{18}$$

Now, assume Airy's stress function $\chi(\xi, \eta, z, t)$ which satisfies condition (10) as

$$\chi = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \overline{f}(q_{n,m}) C_{e_{2n}}(\xi, q_{n,m}) c_{e_{2n}}(\eta, q_{n,m}) \Bigg\{ X_{n,m}$$
$$\times \sinh\left[q_{n,m}\left(z+\frac{h}{2}\right)+\left(z+\frac{h}{2}\right)\right] + zY_{n,m} \cosh\left[q_{n,m}\left(z+\frac{h}{2}\right)+\left(z+\frac{h}{2}\right)\right]\Bigg\} \tag{19}$$
$$\times \exp[(-k\alpha^2{}_{2n,m} + \omega)t] / C_{n,m}$$

in which $X_{n,m}$ and $Y_{n,m}$ are the arbitrary functions that can finally be determined by using condition (9).

$$X_{n,m} = -\frac{G\alpha_t c^2 h^2}{2q_{2n,m}}\left(\frac{1+\nu}{1-\nu}\right),\ Y_{n,m} = 0 \tag{20}$$

By substituting $X_{n,m}$ and $Y_{n,m}$ from equation (20) into equation (19) we get the final Airy's function as

$$\chi = -G\alpha_t \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c^2h^2\left(\frac{1+\nu}{1-\nu}\right)\overline{f}(q_{n,m}) C_{e_{2n}}(\xi, q_{n,m}) c_{e_{2n}}(\eta, q_{n,m})$$
$$\times \sinh\left[q_{n,m}\left(z+\frac{h}{2}\right)+\left(z+\frac{h}{2}\right)\right] \exp[(-k\alpha_{2n,m}^2 + \omega)t] / 2q_{2n,m} C_{n,m} \tag{21}$$

Using equations (8) and (18), one obtains the stresses due to temperature as:

$$\begin{aligned}\bar{\sigma}_{\xi\xi} &= -c^2 G h^4 \cosh 2\xi + \frac{c^4 h^6}{4 q_{2n,m} C_{n,m}} G\alpha_t \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \\ &\times \sin[\tfrac{1}{2}(h+2z)(1+\gamma)]\{(\cos 2\eta) ce''_{2n}(\eta, q_{2n,m}) Ce_{2n}(\xi, q_{2n,m}) \\ &+ ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi - Ce_{2n}(\xi, q_{2n,m}) \\ &\times ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \tag{22}$$

$$\begin{aligned}\bar{\sigma}_{\eta\eta} &= -c^2 G h^4 \cosh 2\xi - \frac{c^4 h^6}{4 q_{2n,m} C_{n,m}} G\alpha_t \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \\ &\times \sin[\tfrac{1}{2}(h+2z)(1+\gamma)]\{(\cos 2\eta) ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \\ &+ ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi - Ce_{2n}(\xi, q_{2n,m}) \\ &\times ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \tag{23}$$

$$\begin{aligned}\bar{\sigma}_{\xi\eta} &= \frac{c^4 h^6}{4 q_{2n,m} C_{n,m}} G\alpha_t \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \\ &\times \sin[\tfrac{1}{2}(h+2z)(1+\gamma)]\{(\cos 2\eta - \cosh 2\xi) ce'_{2n}(\eta, q_{2n,m}) \\ &\times Ce'_{2n}(\xi, q_{2n,m}) + ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \\ &\times \sin 2\eta + Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\xi\} \\ &\times \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \tag{24}$$

Using equations (11) and (21), one obtains the thermal complementary stresses as:

$$\begin{aligned}\bar{\bar{\sigma}}_{\xi\xi} &= -\frac{G\alpha_t c^4 h^6}{4 q_{2n,m} C_{n,m}} \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \sin[\tfrac{1}{2}(h+2z)(1+\gamma)] \\ &\times \{(\cos 2\eta - \cosh 2\xi) ce''_{2n}(\eta, q_{2n,m}) Ce_{2n}(\xi, q_{2n,m}) \\ &+ ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi - Ce_{2n}(\xi, q_{2n,m}) \\ &\times ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{\bar{\sigma}}_{\eta\eta} &= -\frac{G\alpha_t c^4 h^6}{4 q_{2n,m} C_{n,m}} \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \sin[\tfrac{1}{2}(h+2z)(1+\gamma)] \\ &\times \{-(\cos 2\eta - \cosh 2\xi) ce_{2n}(\eta, q_{2n,m}) Ce''_{2n}(\xi, q_{2n,m}) \\ &- ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi \\ &+ Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \\ &\times \exp\{-[k\alpha^2_{2n,m} + \omega]t\}\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{\bar{\sigma}}_{\xi\eta} &= -\frac{G\alpha_t c^4 h^6}{4 q_{2n,m} C_{n,m}} \left(\frac{1+\nu}{1-\nu}\right) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) \sin[\tfrac{1}{2}(h+2z)(1+\gamma)] \\ &\times \{(\cos 2\eta - \cosh 2\xi) ce'_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \\ &+ ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\eta \\ &+ Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\xi\} \\ &\times \exp\{-[k\alpha^2_{2n,m} + \omega]t\}\end{aligned} \tag{28}$$

By substituting equations (22-27) into equation (12) we obtain the complete solution in terms of the displacement potential ϕ and the stress function χ as:

$$\begin{aligned}\sigma_{\xi\xi} &= \frac{G\alpha_t c^4 h^6}{4q_{2n,m} C_{n,m}}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{f}(q_{2n,m})(\sin[\frac{1}{2}(h+2z)(1+\gamma)] \\ &- \sinh[\frac{1}{2}(h+2z)(1+\gamma)])\{-(\cos 2\eta - \cosh 2\xi) ce''_{2n}(\eta, q_{2n,m}) \\ &\times Ce_{2n}(\xi, q_{2n,m}) + ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi \\ &- Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \quad (29)$$

$$\begin{aligned}\sigma_{\eta\eta} &= -\frac{G\alpha_t c^4 h^6}{4q_{2n,m} C_{n,m}}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{f}(q_{2n,m})(\sin[\frac{1}{2}(h+2z)(1+\gamma)] \\ &- \sinh[\frac{1}{2}(h+2z)(1+\gamma)])\{(\cos 2\eta - \cosh 2\xi) ce_{2n}(\eta, q_{2n,m}) \\ &\times Ce''_{2n}(\xi, q_{2n,m}) + ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\xi \\ &- Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\eta\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \quad (30)$$

$$\begin{aligned}\sigma_{\xi\eta} &= \frac{G\alpha_t c^4 h^6}{4q_{2n,m} C_{n,m}}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{f}(q_{2n,m})(\sin[\frac{1}{2}(h+2z)(1+\gamma)] \\ &- \sinh[\frac{1}{2}(h+2z)(1+\gamma)])\{(\cos 2\eta - \cosh 2\xi) ce'_{2n}(\eta, q_{2n,m}) \\ &\times Ce'_{2n}(\xi, q_{2n,m}) + ce_{2n}(\eta, q_{2n,m}) Ce'_{2n}(\xi, q_{2n,m}) \sin 2\eta \\ &+ Ce_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \sin 2\xi\} \exp[-(k\alpha^2_{2n,m} + \omega)t]\end{aligned} \quad (31)$$

4. Transition to circular plate

When the elliptical plate tends to a circular plate of the radius ξ_0, the semi-focal $c \to 0$ and then α_m is the root of the transcendental equation $J_0(\alpha_m) = 0$. Also $e \to 0$ [as $\xi \to \infty$], $\cosh 2\xi\, d\xi \to$ $2\cosh 2\xi \sinh 2\xi\, d\xi \to$ $2r dr / c^2$, $\sinh\xi \to \cosh\xi$, $h\cosh\xi \to r$ [as $h \to 0$], $\cosh\xi\, d\xi \to r\, dr$, $h \sinh\xi\, d\xi \to dr$,

By using results from [12] we obtain:

$$Ce_0(\xi, q_{0,m}) \to p'_0 J_0(\lambda_m r),\ Ce'_0(\xi, q_{0,m}) \to p'_0 J'_0(\lambda_m r),\ Ce''_0(\xi, q_{0,m}) \to p'_0 J''_0(\lambda_m r),\ ce_0(\eta, q_m) \to 1/\sqrt{2},$$

$$A_0^{(0)} \to 1/\sqrt{2}, A_2^{(0)} \to 0,\ \Theta_{2m} \to 0,\ \lambda^2_{0,m} = \alpha^2_{0,m}/a^2 = \alpha^2_m/a^2 = \lambda^2_m,\ p'_0 = Ce_0(0, q_{0,m}) ce_0(2\pi, q_{0,m}) / A_0^{(0)}.$$

Taking into account the parameters above, the temperature distribution in the cylindrical coordinate is finally represented by:

$$T(r,z,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} f_n(t) \sin\left[\lambda_m\left(z + \frac{\ell}{2}\right) + \left(z + \frac{\ell}{2}\right)\right] p'_0 J_0(\lambda_m r) \quad (32)$$

in which

$$f_n(t) = A_{n,m} \exp(-\kappa \lambda_m^2 t),$$

$$A_m = \frac{T_0 \exp(-\omega t) \int_0^a r\, p'_0 J_0(\lambda_m r) f(r)\, dr}{\pi\ell \int_0^a r [p'_0 J_0(\lambda_m r)]^2\, dr}$$

The aforementioned degenerated result agrees with the previous result [13].

5. Numerical Results, Discussion, and Remarks

For the purpose of simplicity of calculation, the following dimensionless values are introduced:

$$\left.\begin{aligned}&\bar{\xi}=\xi/\xi_0,\ \bar{z}=z/\xi_0,\ e=c/\xi_0,\ \tau=\kappa t/\xi_0{}^2,\\&\bar{\theta}=T/T_0,\bar{\sigma}_{ij}=\sigma_{ij}/E\alpha\theta_0\ \ (i,j=\xi,\eta)\end{aligned}\right\} \tag{33}$$

By substituting the value of equation (33) into equation (13) and components of stresses, we obtained the expressions for temperature and the thermal stresses, respectively, for our numerical discussion. The numerical computations have been carried out for an Aluminum (pure) elliptical plate with the physical parameter as $\xi_0 = 1$ m, $\ell = 0.08$ m, reference temperature as 150 0C, and $f(\xi,\eta)=\delta(\xi-\xi_i)\delta(\eta-\eta_i)$ in which $0\le\xi_i\le\xi_0$ and $0\le\eta_i\le 2\pi$. The thermo-mechanical properties are considered as modulus of elasticity E = 70 GPa, Poisson's ratio υ = 0.35, thermal expansion coefficient α = 23×10-6 /0C, thermal diffusivity κ = 84.18 m2s−1, and thermal conductivity λ = 204.2 Wm−1K−1. The $q_{2n,m} = 0.0986$, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292, and 39.4784 are the positive and real roots of the transcendental equation $Ce_{2n}(a,q)=0$. In order to examine the influence of heating on the plate, the numerical calculations were performed for all the variables, and the numerical calculations are depicted in the following figures with the help of MATHEMATICA software. Figures 2–5 illustrate the numerical results of the temperature distribution and stresses of the elliptical plate under the thermal boundary condition that are subjected to arbitrary initial temperature on the upper face while keeping lower face at zero temperature. Figure 2(a) indicates the temperature distribution along the direction of $\bar{z}$ -axis of the plate. The maximum values of temperature magnitude arise from the upper face due to additional heat supply. The distribution of temperature gradient at each instance decreases the axial direction tending and attains its minimum at the lower face which is kept at zero temperature.

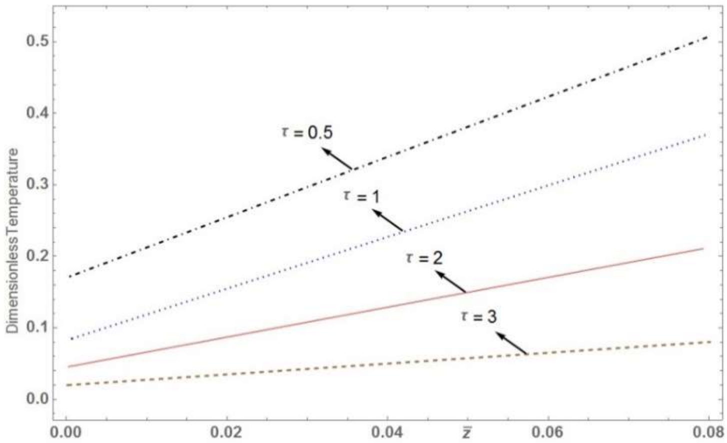

Fig. 2(a). Temperature distribution $\bar{\theta}$ along $\bar{z}$ for different values of τ

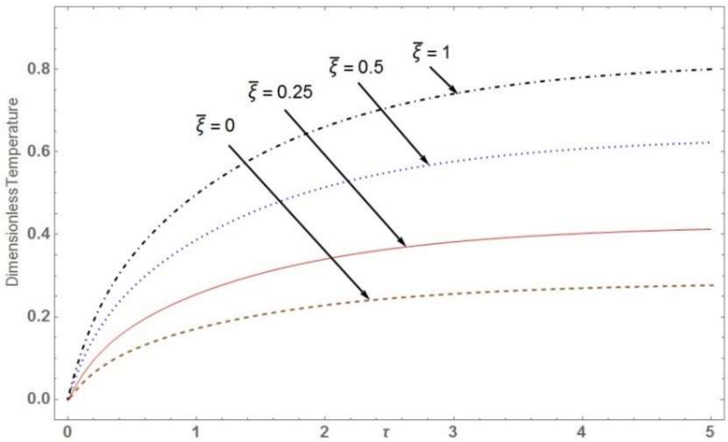

Fig. 2(b). Temperature distribution $\bar{\theta}$ along τ for different values of $\bar{\xi}$

Fig. 2(b) illustrates the temperature profile along the time direction for various values of the radius. Temperature trend increases along the time direction from inner core to the outer curved surface irrespective of the angular direction for a fixed value of $\bar{z}$. At the outer part of the thickness, temperature fluctuation seems more stable due to the accumulation of energy which is caused by more exposure to heat sources; hence, thermal expansion is more at the outer part of the plate which gives high tensile force. Figure 3(a) depicts that the radial stress $\bar{\sigma}_{\xi\xi}$ attains its minimum at the outer core due to the compressive forces occurring at the outer region. Figure 3(b) indicates the dimensionless radial stress along the angular direction for different values of the thickness; it is clear that the radial stress follows the sinusoidal nature due to the periodicity of Mathieu function.

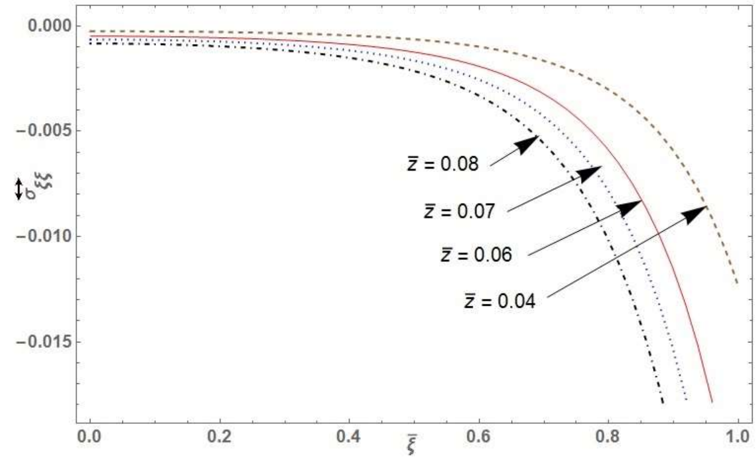

Fig. 3(a). Dimensionless radial stress along $\bar{\xi}$ for different values of $\bar{z}$

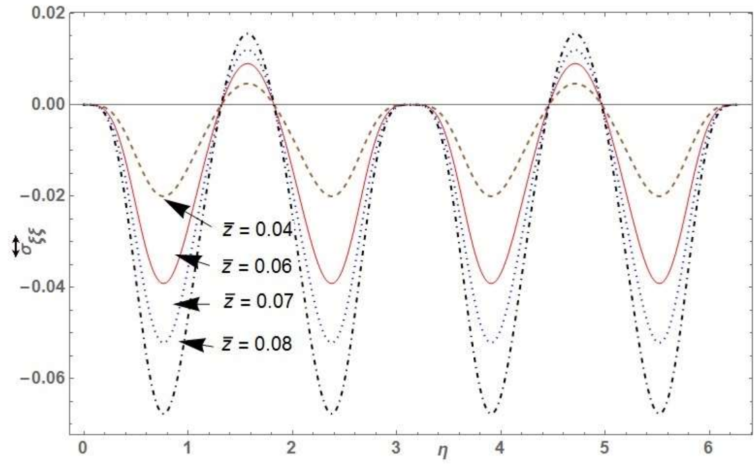

Fig. 3(b). Dimensionless radial stress along η for various values of $\bar{z}$

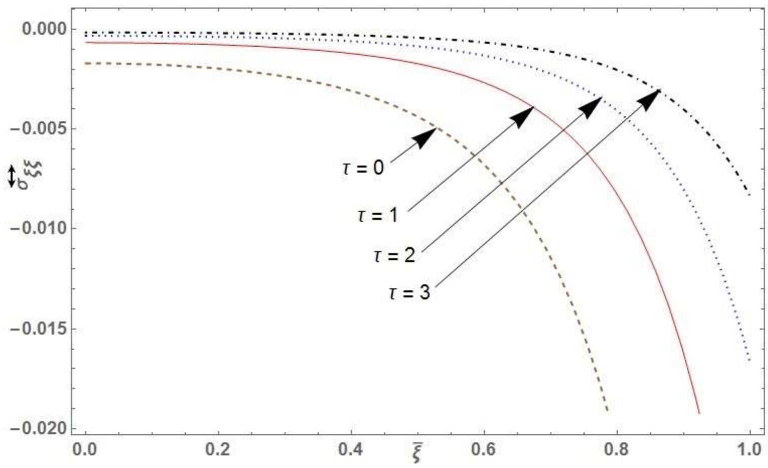

Fig. 3(c). Dimensionless radial stress along $\bar{\xi}$ for different values of τ

Figure 3(c) shows the dimensionless radial stress along the radial direction for different time series; the stress attains its maximum at the core of the inner part which may be due to initial temperature, and lowers towards the outer part for different time series along the radial direction. Figure 3(d) depicts the dimensionless radial stress along the time series for different radii;

the stress increases gradually with the increase in time for different radii and stabilizes after a certain time due to the rate of temperature change with respect to time.

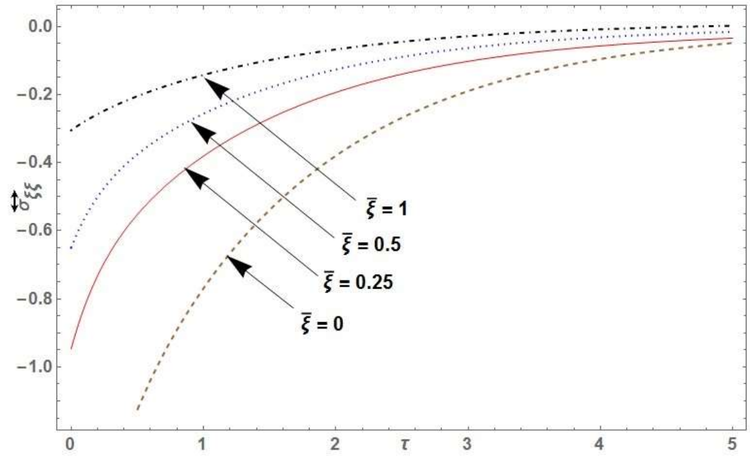

Fig. 3(d). Dimensionless radial stress along τ for different values of $\bar{\xi}$

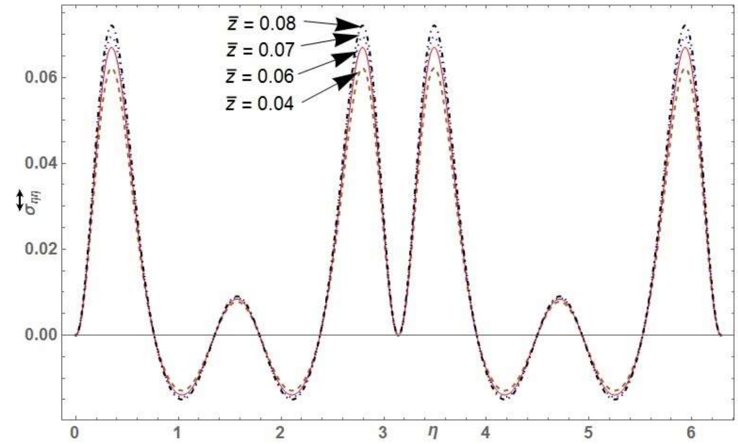

Fig. 4(a). Dimensionless tangential stress along η for different values of $\bar{z}$

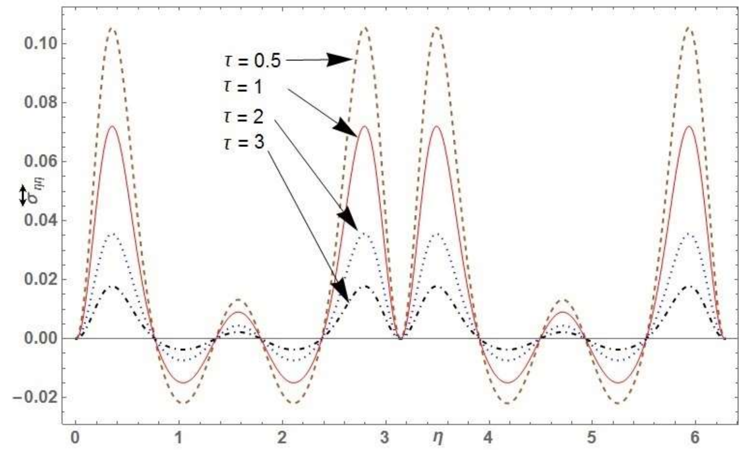

Fig. 4(b). Dimensionless tangential stress along η for different values of τ

Figures 4(a) and 4(b) represent dimensionless tangential stress; both graphs are sinusoidal in nature showing vibration of the plate due to the periodicity of Mathieu function. Figure 4(c) gives the dimensionless tangential stress along time series for different radii; it is shown that tangential stresses are maximum at inner core of the plate and lower towards the outer end and stabilize after a certain time.

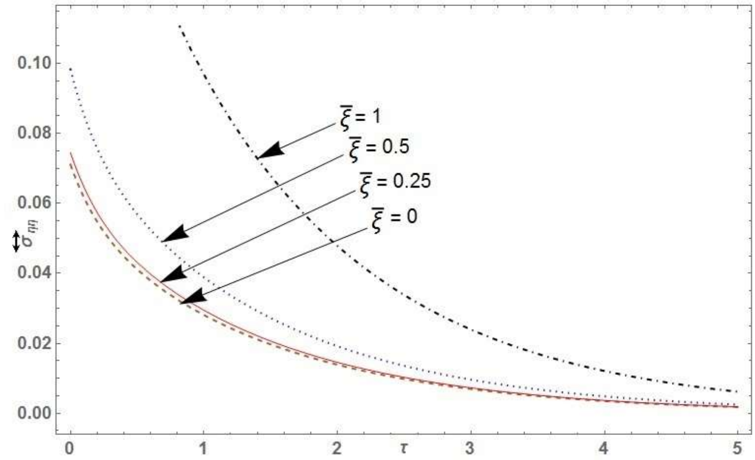

Fig. 4(c). Dimensionless tangential stress along τ for different values of $\bar{\xi}$

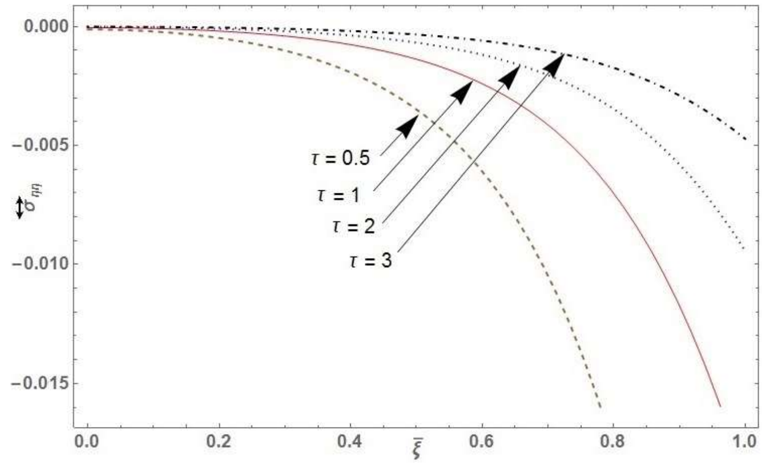

Fig. 4(d). Dimensionless tangential stress along $\bar{\xi}$ for different values of τ

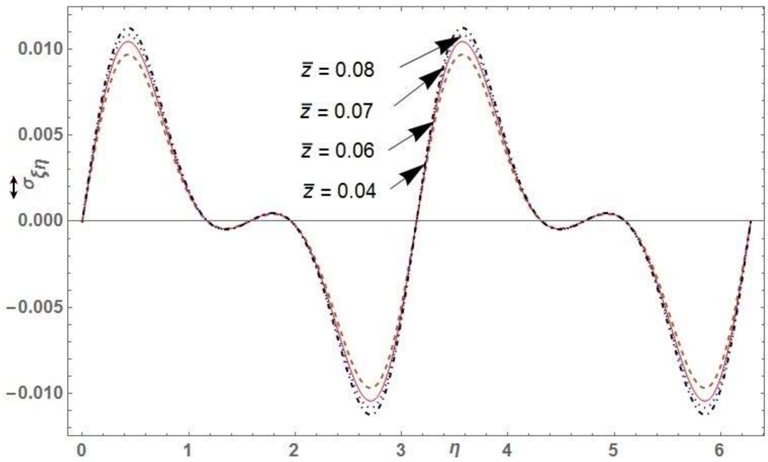

Fig. 5(a). Shear stress $\ddot{\sigma}_{\xi\eta}$ along η for different values of $\bar{z}$

Figure 5(a) represents dimensionless shear stress profile. The graph is sinusoidal in nature showing vibration of the plate due to the periodicity of Mathieu function. Figures 5(b) and 5(c) indicate the shear stress along $\bar{\xi}$ - direction of the plate for different values of $\bar{z}$ and time. The maximum value of stress magnitude occurs at the outer edge due to the additional heat energy throughout the body. The distribution of stress at every point of $\bar{z}$ and time decreases towards the central part of the ellipse boundary; thus, it tends to zero at the unheated part. Figure 5(d) depicts that the shear stress $\ddot{\sigma}_{\xi\eta}$ attains zero at $\eta = 0, \pi/2, \pi, 3\pi/2, 2\pi$, whereas on the other parts, it attains its maximum due to the accumulation of thermal energy dissipated by sectional.

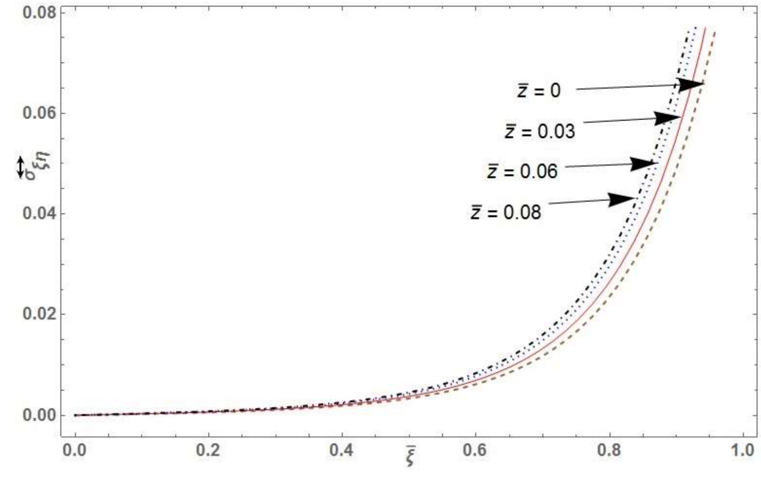

Fig. 5(b). Shear stress $\ddot{\sigma}_{\xi\eta}$ along $\bar{\xi}$ for various values of $\bar{z}$

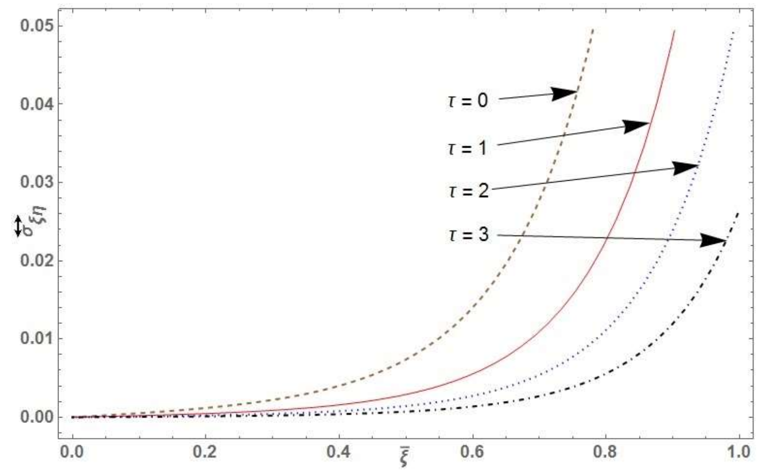

Fig. 5(c). Shear stress $\ddot{\sigma}_{\xi\eta}$ along $\bar{\xi}$ for different values of τ

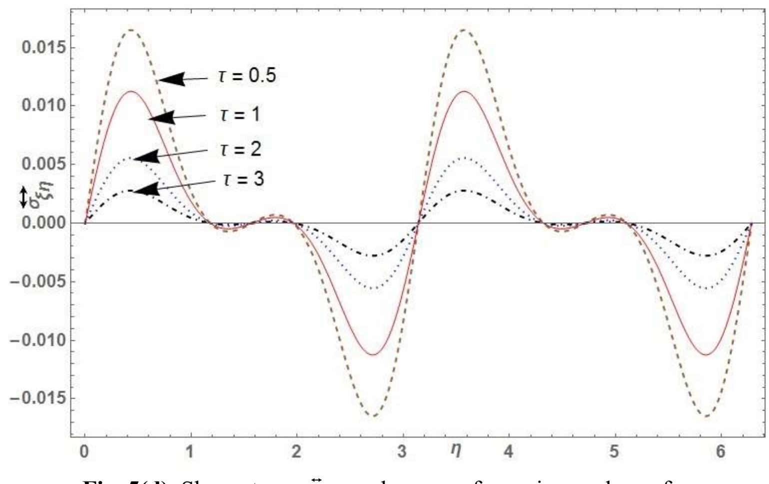

Fig. 5(d). Shear stress $\ddot{\sigma}_{\xi\eta}$ along η for various values of τ

6. Conclusion

In this manuscript, we have described the theoretical treatment of the quasi-static thermal stresses in a thin elliptical plate. The temperature distribution and the stresses in the form of ordinary and the modified Mathieu functions are used to determine the solution by classical methods. The analytical technique proposed here is relatively straightforward and widely applicable compared with the methods proposed by other researchers. The results obtained while carrying out the research are generalized as follows:

- The advantage of this approach is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions during induced stresses under thermal loading.

- The maximum tensile stress shifts from the outer surface due to maximum expansion of the outer part of the plate, and its absolute value increases with the radius. This shifting of stress could be due to heat, stresses, concentration, or available internal heat sources under the known temperature field.
- Finally, the maximum tensile stress occurs in the circular core on the major axis compared with the elliptical core indicating the weak distribution of heat. This difference might be due to insufficient penetration of heat through the elliptical inner surface.
- The aforementioned thermal stress calculation concept can be beneficial in the field of micro-devices or microsystem applications, planar continuum robots, prediction of the elastoplastic bending, and so forth.

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QUASI-STATIC TRANSIENT THERMAL STRESSES IN A THICK ANNULAR PLATE SUBJECTED TO SECTIONAL HEAT SUPPLY

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ABSTRACT: The principal aim of this paper is to investigate the thermoelastic problems in a thick annular plate subjected to sectional heat supply on the upper surfaces whereas the fixed circular edges are at zero temperature. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically.
Keywords: thick annular plate, thermoelasticity, integral transform

1. INTRODUCTION

As a result of the increased usage of industrial and construction materials the interest in the thermal stress problems has grown considerably, typified by the annular fins of heat exchangers and brake disc rotors, because of its elementary geometry. Therefore, a number of theoretical studies concerning them have been reported so far. For example, Nowacki [6] has determined steady-state thermal stresses in circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge. The direct thermoelastic problem in an annular fin is studied by Wu [10] investigates the transient thermal stresses in an annular fin with its base subjected to a heat flux of a decayed exponential function of time. Wankhede [11] has determined the quasi-static thermal stresses in thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Gogulwar and Deshmukh [3] solved the inverse problem of thermal stresses in a thin annular disc, which was further generalized [2] in direct problem. Chiu and Chen [1] investigated stress-field in an annular fin of temperature-dependent conductivity under a periodic heat transfer boundary condition is analyzed by the Adomian's decomposition method. Recently Ootao et al. [8] performed analysis of a three-dimensional transient thermal stress problem is developed for a nonhomogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and /or outer surfaces. In this paper, our attempt has been made to discuss quasi-static transient thermal stresses in a thick annular plate $a \le r \le b$, $-h \le z \le h$ and the result illustrated numerically and graphically by using integral transform technique. No one previously studied such type of problem. This is a new contribution to the field.

2. FORMULATION OF THE PROBLEM

Consider a thick annular plate of thickness 2h, occupying a space D defined by $a \le r \le b$, $-h \le z \le h$. Let the plate be subjected to a transient asymmetric temperature field on the axial direction & axisymmetric temperature field on the radial direction of the cylindrical coordinate system. Initially the plate is kept at zero temperature the arbitrary heat flux $Qf(r)/\lambda$ is prescribed over the upper surface (z = h) and the lower surface (z = -h) the fixed circular edge (r = a and r = b) are at zero temperature. Assume the upper and lower surface of thick annular plate are traction–free surface under this realistic prescribed condition the quasi-static transient thermal stresses are required to be determined.

2.1. Temperature distribution

The transient heat conduction equation is given as follows

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

in which thermal diffusivity of the material of the plate is denoted as $\kappa=\lambda/\rho C$, λ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant, subjected to the initial and boundary conditions as

$$T=0 \text{ at } t=0 \tag{2}$$

$$T=0 \text{ at } r=a,\ -h\le z\le h\ ,\ t>0 \tag{3}$$

$$T=0 \text{ at } r=b,\ -h\le z\le h\ ,\ t>0 \tag{4}$$

$$T=0 \text{ at } z=-h,\ a\le r\le b\ ,\ t>0 \tag{5}$$

$$\frac{\partial T}{\partial z}=(Q/\lambda)f(r)t, \text{ at } z=h,\ a\le r\le b\ ,\ t>0 \tag{6}$$

2.2. Thermal displacements and thermal stress

The Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [5]

$$\begin{aligned}&\nabla^2 u_r-\frac{u_r}{r}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial \theta}{\partial r}=0\\&\nabla^2 u_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial \theta}{\partial z}=0\end{aligned} \tag{7}$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential ϕ and Love's function L as [4]

$$u_r=\frac{\partial \phi}{\partial r}-\frac{\partial^2 L}{\partial r\partial z}, u_z=\frac{\partial \phi}{\partial z}+2(1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial^2 z} \tag{8}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2\phi=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t T \tag{9}$$

and the Love's function L must satisfy the equation

$$\nabla^2(\nabla^2 L)=0 \tag{10}$$

in which $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$

The component of the stresses are represented by the use of the potential ϕ and Love's function L as

$$\begin{aligned}&\sigma_{rr}=2G\left\{\left(\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{\partial^2 L}{\partial r^2}\right)\right\}, \sigma_{\theta\theta}=2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{1}{r}\frac{\partial L}{\partial r}\right)\right\},\\&\sigma_{zz}=2G\left\{\left(\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial z^2}\right)\right\}, \sigma_{rz}=2G\left\{\frac{\partial^2\phi}{\partial r\partial z}+\frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial z^2}\right)\right\}\end{aligned} \tag{11}$$

in which G and υ are the shear modulus and Poisson's ratio respectively.

The boundary condition on the traction free surface stress functions are

$$\sigma_{rr}=\sigma_{rz}=0 \text{ at } z=\pm h \tag{12}$$

Equations (1) to (16) constitute the mathematical formulation of the problem.

3. SOLUTION OF THE PROBLEM

3.1. Solution for Temperature distribution

Applying Laplace transformation [9] of the equation (1) to (6) with respect to t and using the equation (2) one obtain

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} + \frac{\partial^2 \bar{T}}{\partial z^2} = \frac{P}{K}\bar{T} \tag{13}$$

with boundary condition

$$\frac{\partial \bar{T}}{\partial z} = \frac{Qf(x)}{\lambda P^2} \quad \text{at} z = h \tag{14}$$

$$\bar{T} = 0 \quad \text{at } z = -h \tag{15}$$

$$\bar{T} = 0 \quad \text{at} r = a \ \text{and} r = b \tag{16}$$

where p is Laplace transform parameter and $\bar{T}$ Laplace transform of T

Introducing theHankel transform over the variable r and its inverse transformation defined [7] as

$$\bar{\bar{T}}(\alpha_m, z, p) = \int_a^b r K_0(\alpha_m r)\bar{T}(r,z,p)\,dr,$$
$$\bar{T}(r,z,p) = \sum_{n=1}^{\infty} \bar{\bar{T}}(\alpha_m, z, p) K_0(\alpha_m r) \tag{17}$$

in which $K_0(\alpha_m r) = \frac{R_0(\alpha_m r)}{\sqrt{N}}, R_0(\alpha_m, r) = \frac{J_0(\alpha_m r)}{J_0(\alpha_m b)} - \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)}, N = (b^2/2)R_0'^2(\alpha_m b) - (a^2/2)R_0'^2(\alpha_m a),$

and α_1, α_2..are roots of the transcendental equation $R_0(\alpha_m, a) = 0$ with $J_n(x)$ is the Bessel function of the first kind of order n and $Y_n(x)$ is the Bessel function of the second kind of order n.

Applying the finite Hankel integral transform, and its inversion theorems for both transforms, yield

$$T = \left(\frac{Q}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)}{2h\sqrt{N}}\right)\wp_n R_0(\alpha_m, r) \tag{18}$$

in which $q^2 = \alpha_m^2 + P/K$, $\hbar_m = \alpha_m^2 + (2n+1)^2\pi^2/16h^2$,

$$\wp_n = \sin\left[(z+h)\frac{(2n+1)\pi}{4h}\right]\int_0^t \tau \exp\{-k\hbar_m(t-\tau)\}d\tau / \sin\left[(2n+1)\frac{\pi}{2}\right]$$

and $\bar{f}(\alpha_m)$ is the Hankel transform of $f(r)$.

3.2. Solution for thermal stresses

(a) Goodier thermoelastic displacement potential ϕ.

Referring to the fundamental equation (1) and its solution (18) for the heat conduction problem, the solution for the displacement function are represented by the Goodier's thermoelastic displacement potential ϕ governed by equation (9) are represented by

$$\phi = -\left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)}{\sqrt{N}}\right)\wp_{n\,\hbar_m} R_0(\alpha_m, r) \tag{24}$$

(b) Love's function L

Similarly, the solution for Love's function L are assumed so as to satisfy the governed condition of equation (12) as

$$L = \left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)\wp\,\alpha_{mn}}{\sqrt{N}}\right)R_0(\alpha_m, r)[H_{mn}\cosh[\alpha_m(z+h)] + R_{mn}\alpha_m(z+h)\sinh[\alpha_m(z+h)] \tag{25}$$

in which H_{mn} and R_{mn} are arbitrary unknown functions

(c) Displacement and Thermal stresses

In this manner, two displacement functions in the cylindrical coordinate system ϕ and L are fully formulated. Now, in order to obtain the displacement components, we substitute the values of thermoelastic displacement potential ϕ and Love's function L in equations (9) and (10), one obtains

$$U_r = \left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0'(\alpha_m, r)\{\wp_n / \hbar_m - [\alpha_m^2 H_{mn}\sinh[\alpha_m(z+h)] + R_{mn}\alpha_m^3 (z+h)\cosh[\alpha_m(z+h)]\} \quad (26)$$

$$U_z = -\left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0(\alpha_m, r)\{(2n+1)\wp_n \pi \cot\left[(z+h)\frac{(2n+1)\pi}{4h}\right] / 4h\hbar_m$$
$$-(\alpha_m^2 H_{mn}\cosh[\alpha_m(z+h)]) - R_{mn}[(z+h)\alpha_m^3 \sinh[\alpha_m(z+h)] - 2(1-2\upsilon)] \quad (38)$$
$$\times (z+h)\alpha_m^2 \cosh[\alpha_m(z+h)]]\}$$

$$\sigma_{rr} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0''(\alpha_m, r) - \frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0(\alpha_m, r)\right.$$
$$+ H_{mn}\left\langle R_0'(\alpha_m, r)\alpha_m \cosh[\alpha_m(z+h)]\right\rangle + R_{mn}\left\langle R_0'(\alpha_m, r) 2\alpha_m^3 \sinh[\alpha_m(z+h)]\right. \quad (39)$$
$$\left.\left. + R_0''(\alpha_m, r)[\alpha_m^2 (z+h)\cosh[\alpha_m(z+h)] + \sinh[\alpha_m(z+h)]\right\rangle\right\}$$

$$\sigma_{\theta\theta} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0'(\alpha_m, r) - \frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0(\alpha_m, r)\right.$$
$$+ H_{mn}\alpha_m^2\left\langle R_0'(\alpha_m, r)\sinh[\alpha_m(z+h)]\right\rangle / r + R_{mn}\alpha_m^2\left\langle 2\upsilon\alpha_m R_0(\alpha_m, r)\sinh[\alpha_m(z+h)]\right. \quad (40)$$
$$\left.\left. + r^{-1} R_0'(\alpha_m, r)(\sinh[\alpha_m(z+h)] + \alpha_m(z+h)\cosh[\alpha_m(z+h)])\right\rangle\right\}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}} R_0(\alpha_m, r) - \left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}} R_0(\alpha_m, r)\right. \quad (41)$$
$$\left. - \alpha_m^3 H_{mn}\sinh[\alpha_m(z+h)] + \alpha_m^3 R_{mn}\left\langle (1-2\upsilon)\sinh[\alpha_m(z+h)] - \alpha_m(z+h)\cosh[\alpha_m(z+h)]\right\rangle\right\}$$

$$\sigma_{rz} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0'(\alpha_m, r) \times \frac{(2n+1)\pi\alpha_m}{4h\hbar_m}\cot\left[(z+h)\frac{(2n+1)\pi}{4h}\right] + H_{mn}\alpha_m^3\cosh[\alpha_m(z+h)]\right. \quad (42)$$
$$\left. + R_{mn}(2\upsilon\cosh[\alpha_m(z+h)] + \alpha_m(z+h)\sinh[\alpha_m(z+h)])\right\}$$

(d) Determination of unknown arbitrary function H_{mn} and R_{mn}

Applying boundary condition (17) to the equation (25) and (28) one obtains

$$H_{mn} = 0 \quad (43)$$

$$R_{mn} = \frac{(2n+1)\pi}{4h\alpha_m^2 2\upsilon\hbar_m}\frac{\wp_n}{\sin\left[(z+h)\frac{(2n+1)\pi}{4h}\right]} \quad (44)$$

4. SPECIAL CASE AND NUMERICAL CALCULATIONS

Setting

$$f(r) = (r^2 - a^2)(r^2 - b^2) \quad (45)$$

Applying finite Hankel transform as defined in equation (21) to the equation (45), one obtain

$$\bar{f}(\alpha_m) = \frac{8\{(a^2\alpha_m^2 - 3b^2\alpha_m^2 + 16)J_0(\alpha_m a) - (b^2\alpha_m^2 - 3a^2\alpha_m^2 + 16)J_0(\alpha_m b)\}}{\pi\sqrt{N}\alpha_m^6 J_0(\alpha_m a)J_0(\alpha_m b)Y_0(\alpha_m b)} \quad (46)$$

5. NUMERICAL CALCULATIONS

The numerical calculation have been carried out for (SN 50C) plate with the parameters a = 1m, b = 2m, h = 0.3m, thermal diffusivity $k = 15.9 * 10^{-6}(m^2 s^{-1})$ and Poisson ratio v = 0.281 with $\alpha_1 = 3.120, \alpha_2 = 6.2734, \alpha_3 = 9.4182, \alpha_4 = 12.5614, \alpha_5 = 15.7040$ being the Positive roots of transcendental equation $R_0(\alpha_m, a) = 0$ For convenience setting $A = QK/\pi\lambda 10^5$, $B = 2GQK/\pi\lambda 10^5$ in the expression (3.39) to (3.44) The numerical expression for temperature, displacement and stress components are obtained by equations (34) and (37) to (42).In order to examine the influence of heat flux on the upper and lower surface of thick plate, one performed the numerical calculations r = 1, 1.2, 1.4, 1.6, 1.8, 2 m and z =- 0.3, -0.15, 0, 0.15, 0.3 m. and t = 5, Numerical variations in radial and axial directions are shown in the figures.

6. CONCLUDING REMARKS

In this study, we have treated thermoelastic problem of a thick annular plate which is considered traction free. We successfully established and obtained the expressions for temperature distribution, displacement and stress function due asymmetric arbitrary heat flux. Then, in order to examine the validity of boundary value problem, we analyze, as a particular case with mathematical model for $f(r) = (r^2 - a^2)(r^2 - b^2)$ and numerical calculations were carried out. The thermoelastic behavior is examined such as temperature, displacement and stresses with the help of arbitrary heat flux at upper surface applied.

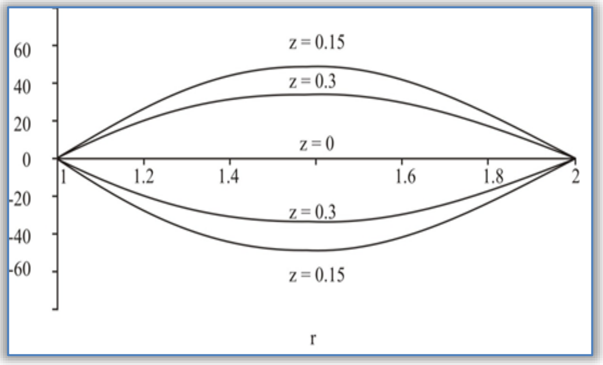

Figure 1: Axial displacement profile along axial direction

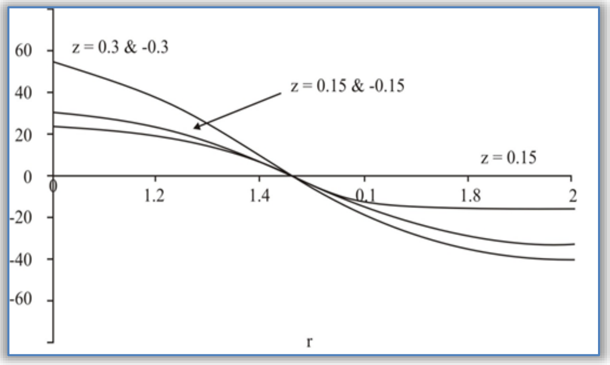

Figure 2: Radial displacement profile along axial direction

Figure 1 shows the axial displacement u_z occurs at the center i.e. r = 1.5 in radial direction where as in radial direction decreases from lower surface to upper surface.

As shown in Figure 2 the variation of thermal stress in the radial displacement u_r decreases from inner circular surface to outer circular surface in radial direction where as in axial direction it take place at upper and lower surfaces of the plate.

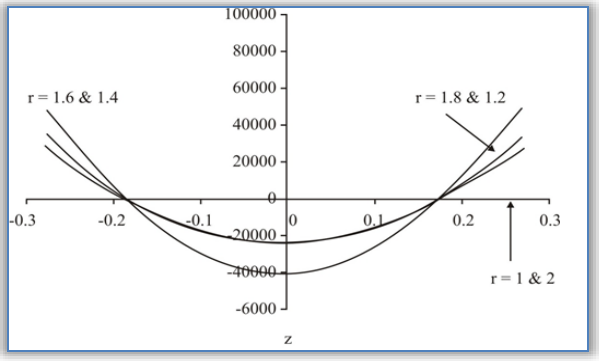

Figure 3: Radial stress distribution along radial direction

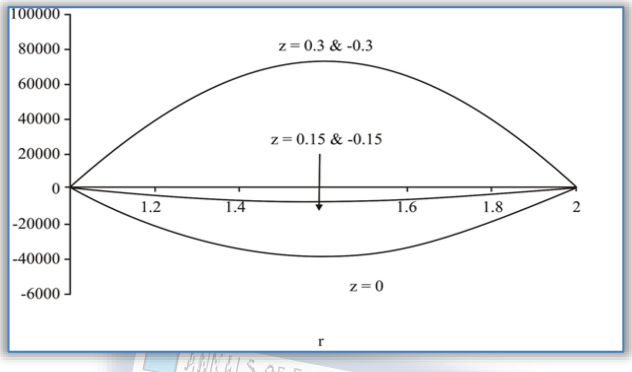

Figure 4: Radial stress distribution along axial direction

Figure 3 and 4 shows the radial stress function σ_{rr} develops tensile stress at upper and lower surface of the plate, where as it develop compressive stress in the middle of plate.

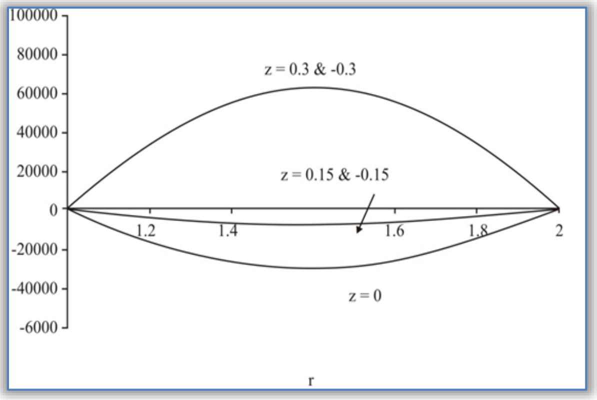

Figure 5: Tangential stress distribution along radial direction

Figure 5 shows the variation of the stress function $\sigma_{\theta\theta}$develops tensile stress at the upper and lower surface of the plate where as it develops compressive stress in the middle of plate. We may conclude that the system of equations proposed in this study can be adapted to design of useful structures or machines in engineering applications in the determination of thermoelastic behavior at every instant and at all points of thick annular disc of finite height.

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ANNALS of Faculty Engineering Hunedoara
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Report of the work done

1) Brief Objective of Minor Research Project

The main Object Of the project is: -

- ✓ To formulate the heat conduction problems on various bodies and study their thermo-elasticity.
- ✓ Analysis of the problems
- ✓ Study of existing method
- ✓ Construct new methods
- ✓ Application related with engineering field.
- ✓ The main emphasis, in the present project will be on the analytical and numerical approach for investigating, problems arising in thermo-elasticity.
- ✓ As special case I construct the mathematical models for various bodies and study their thermo-elasticity.

2) Work done so far & Result achieved & Publication

As per the objective of minor research project initially I consider the three problems on Thermo-elasticity Deformation and examine the results numerically by considering the special cases.

❖ **First Research Problem: -**

Inverse quasi–static unsteady–state thermal stresses in a thick circular plate.

The principal aim of this problem is to investigate the thermoelastic problems of a thick circular plate defined as $0 \le r \le a,\ -h \le z \le h$ subjected to the arbitrary heat supply at interior point while circular edge of the thick circular plate at the outer surface and at the lower surface are maintained at zero temperature.. **The governing heat conduction equation has been solved by** using finite Hankel and Laplace integral transform techniques. Goodier's and Michell's functions are used to obtain the displacement components & its associated stresses. The results are obtained in a form in terms of Bessel's function. The results for unknown temperature, displacement, and stresses have been computed numerically considering special functions and illustrated graphically.

Key words and phrases: **thick circular plate, thermo-elasticity, unsteady-state, integral transform**

AMS Subject classifications: 35B07; 35G30; 35K05: 44A10

1) Introduction-

As a result of the increased usage of industrial and construction materials, the interest in the inverse thermal stress problems have grown considerably, typified by main shaft of lathe and the role of the rolling mill, due to the elementary geometry involved. As a result of this, a number of theoretical studies concerning them have been reported so far. However, to simplify this, almost all the studies were conducted on the assumption that the upper and lower surfaces of the circular are insulated or that the heat is dissipated with uniform heat transfer coefficients throughout the surfaces as direct problems. Ashida et al. (2002) emphasized on the inverse transient thermoelastic problem for a composite circular disc. Yang at al. (2002) studied inverse boundary value problem of coupled thermo-elasticity in an infinitely long annular cylinder using simulated exact and inexact measurements. Patil and Krishna (2013) studied inverse steady-state thermoelastic problem of a thin rectangular plate occupying using operational methods. From the previous literatures regarding thick plate as considered, it was observed by the author that no analytical procedure has been established for thick circular plate, considering inverse quasi-static thermoelastic analysis.

In this problem, we consider some new interesting results of the inverse heat conduction problem of thick circular plate occupying the space $D = \{(x, y, z) \in R^3 : 0 \le (x^2 + y^2)^{1/2} \le a, -h \le z \le h\}$, where $r = (x^2 + y^2)^{1/2}$. In a condition wherein a thick circular plate is subjected to arbitrary heat supply at interior point while the circular edge of the thick circular plate at the outer surface and at the lower surface are maintained at zero temperature, the governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel's functions. The mathematical model of final thick circular plate has been constructed with the help of numerical illustrations.

1. Transient Heat Conduction Problem

Consider a thick circular plate of thickness $2h$ occupying space D defined by $0 \le r \le a, -h \le z \le h$, as shown in Figure 1. Let the plate be subjected to an arbitrary known interior temperature $f(r, t)$ within the region $-h \le z \le h$. With lower surface and circular surface $r = a$ at zero temperature. Under this more realistic prescribed condition, the unknown temperature $g(r, t)$ which is at the upper surface of the plate $z = h$ and quasi–static thermal stresses due to unknown temperature $g(r, t)$ is to be determined.

2. Temperature distribution

1. The transient heat conduction equation of the plate is given as follows

2. $$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t} \quad (1)$$

3. where κ is the thermal diffusivity of the material of the disc (which is assumed to be constant), subjected to the initial and boundary conditions

4. $T = 0$ at $t = 0$ (2)

5. $T = 0$ at $r = a$ (3)

6. $T = 0$ at $z = -h$ (4)

7. $T = f(r, t)$ at $z = \xi$ (5)

8. $T = g(r, t)$ at $z = h \quad 0 \le r \le a$ (Unknown) (6)

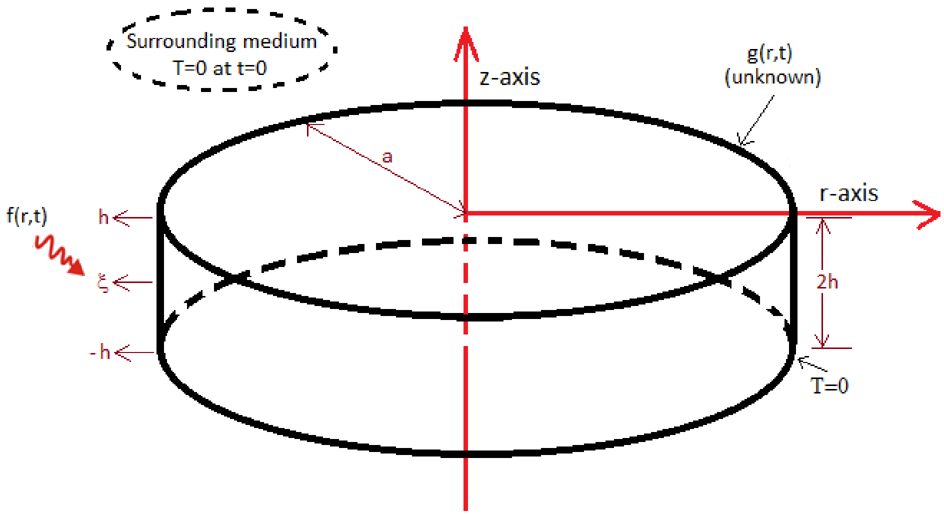

Figure 1. Geometrical configuration of the problem

3. Thermal displacements and thermal stress

Following Noda et al. (2003), we assume that the Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem as

$$\nabla^2 U_r-\frac{U_r}{r}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial T}{\partial r}=0$$
$$\nabla^2 U_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial T}{\partial z}=0 \quad (7)$$

where U_r and U_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e=\frac{\partial U_r}{\partial r}+\frac{U_r}{r}+\frac{\partial U_z}{\partial z} \quad (8)$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential ϕ and Michel's function M

$$U_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z}, \tag{9}$$

$$U_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 M - \frac{\partial^2 M}{\partial^2 z} \tag{10}$$

in which Goodier's thermoelastic potential must satisfy

$$\nabla^2 \phi = K\tau \text{ with } \phi = 0 \text{ at } t = 0 . \tag{11}$$

and the Michel's function *M* must satisfy

$$\nabla^2 \nabla^2 M = 0 \tag{12}$$

in which, *K* is the restraint coefficient and temperature change $\tau = T - T_{i,}\ T_i$ is the initial temperature, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} .$$

The component of the stresses are represented as

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \varphi}{\partial r^2} - k\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{13}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \varphi}{\partial r} - k\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{14}$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \phi}{\partial z^2} - k\tau + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{15}$$

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{16}$$

where *G* and ν are Shear modulus and Poisson's ratio respectively for the traction free surface, the stress function

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } r = a \tag{17}$$

Equations (1) to (17) constitute the mathematical formulation of the problem.

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- ✓ ***<u>A copy of full-page paper is attached. ANNEX</u>***

Second Research Problem

4. Quasi-static transient thermal stresses in an elliptical plate due to sectional heat supply on the curved surfaces over the upper face.

The paper is an attempt to determine quasi-static thermal stresses, in a thin elliptical plate which is subjected to transient temperature on the top face with zero temperature on the lower face and the homogeneous boundary condition of the third kind on the fixed elliptical curved surface. The solution to conductivity equation is elucidated by employing a classical method. The solution of stress components is achieved by using Goodier's and Airy's potential function, involving the Mathieu and modified functions and their derivatives. The numerical results obtained are accurate enough for practical purposes for the better understanding the underlying elliptic object and for the better estimates of the thermal effect of the thermoelastic problem. Conclusions emphasise the importance of better understanding the underlying elliptic structure, improved understanding of its relationship to circular object profile, and better estimates of the thermal effect of the thermoelastic problem.

Keywords: Elliptical plate; temperature distribution; thermal stresses; Mathieu function.

1. Introduction

The theoretical study of the heat flow within hollow elliptical structures are of considerable practical importance in a wide range of sectors such as mechanical, aerospace and food engineering fields for the past few decades. Unfortunately, there are only a few studies concerned with steady and transient state heat conduction problems in elliptical objects. A short history of the research work associated with the thermoelastic insights various approximate methods like the Ritz energy method, Galerkin's Method, finite element models and perturbation theory to solve the system. Of most recent literature, some authors have undertaken the work on heat conduction analysis, which can be summarised as given below. Gupta introduced a finite transform involving Mathieu functions and used for obtaining the solutions of boundary value problem involving elliptic cylinders. Sato subsequently obtained heat conduction problem of an infinite elliptical cylinder during heating and cooling considering the effect of the surface resistance. Recently El Dhaba used boundary integral method to solve the problem of the plane, uncoupled linear thermoelasticity with heat sources for an infinite cylinder with elliptical cross section, subjected to a uniform pressure and a thermal radiation condition on its boundary. However, there aren't many investigations done or studied to eliminate thermoelastic problems successfully. Most recently, Helsing formulated an elastic problem with mixed boundary conditions, that is, Dirichlet conditions on parts of the boundary and Neumann conditions and solved on an interior planar domain using an integral equation method. Dang and Mai estimated mixed boundary value problem for a biharmonic equation of the Airy stress function which models a crack problem of a solid elastic plate using an iterative method. Al Duhaim et al. determined the thermal stress of a mixed boundary value problem in half space using the Jones's modification of the so-called Wiener-Hopf technique. Parnell et al. employed Wiener-Hopf and Cagniard-de Hoop techniques to solve a range of transient thermal mixed boundary value problems in the half space. Nuruddeen and Zaman obtained the analytical solution of transient heat conduction in a solid homogeneous infinite circular cylinder using the Wiener-Hopf technique owing to the mixed nature of the boundary conditions. Very recently, Bhad has obtained few thermoelastic solution for elliptical objects using integral transform technique. The above reviews clearly suggest that, in contrast with the classical circular or rectangular structures case, nearly all investigators so far focused on thermoelastic problems in elliptical membranes either in steady or unsteady state. In particular, there seem to be no rigorous analytical or numerical reports on the quasi-static transient response of a thin elliptical plate subjected to thermal load. The primary purpose of the current work is to fill this gap. The first novelty in our work is that we consider the variational hemivariational inequality defined on a bounded interval of time. The second novelty related to the special structure of the variational-hemivariational inequality which we consider.

The object of this paper is to study the quasi-static thermal stresses in a thin elliptical plate subjected to sectional heat supply on the upper face with the lower face is kept at zero temperature. To establish the quasi-static problem formulation, the following assumptions need to be made (i) The material of the cylinder is elastic, homogeneous, and isotropic, (ii) Thin walled cylinder has been considered during the investigation with a ratio of length to the thickness greater than 8, (iii) The deflection (the normal component of the displacement vector) of the mid-plane is small as compared to the thickness of the plate, and (iv) The stress perpendicular to the middle plane is small compared to the other stress components and may be neglected in the stress-strain relations.

The success of this research mainly lies with the analytical procedures which present a much simpler approach for optimisation of the design regarding material usage and performance in engineering problem, particularly in the determination of thermoelastic behaviour in elliptical disc engaged as the foundation of pressure vessels, furnaces, etc. Actually, by considering a circle as a special kind of ellipse, it is shown that the temperature distribution and history in a circular solution can be derived as a special case of the present mathematical solution for the elliptical disc.

2. Formulation of the problem

It is assumed that a thin elliptical plate is occupying the space $D:\{(\xi,\eta,z)\in R^3 : 0<\xi<\xi_0,\ 0<\eta<2\pi,\ -\ell/2<z<\ell/2\}$ under unsteady-state temperature field with no internal heat source within it. The geometry of the plate as shown in figure 1 indicates that an elliptic coordinate system (ξ,η,z) is the most appropriate choices of the reference frame, which are related to the rectangular coordinate system (x,y,z) by the relation $x=c\cosh\xi\cos\eta,\ \ y=c\sinh\xi\sin\eta,\ z=z$. The curves $\eta=$constant represent a family of confocal hyperbolas while the curves $\xi=$constant constitute a family of confocal ellipses (refer Fig. 1). Both sets of curves intersect each other orthogonally at every point in space.

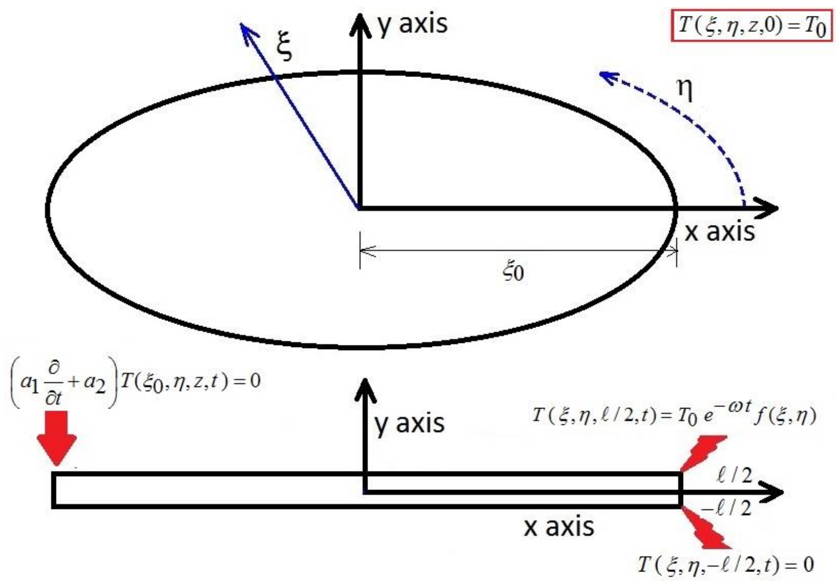

Figure 1. Physical configuration of elliptical plate

The geometry parameters are given as $\xi\in[0,\xi_0]$, $\eta\in[0,2\pi)$ and $z\in[-\ell/2,\ell/2]$. Let the plate be subjected to the arbitrary initial temperature over the upper surface $(z=\ell/2)$ with the lower surface $(z=-\ell/2)$ at zero temperature and boundary condition of the third kind on the curved surface; the quasi-static thermal stresses are required to be determined.

2.1 Heat conduction of the problem

The governing differential equation for heat conduction and boundary conditions can be defined as

$$\frac{1}{\kappa}\frac{\partial}{\partial t}(\xi,\eta,z,t)=h^2\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)T(\xi,\eta,z,t)+\frac{\partial^2}{\partial z^2}T(\xi,\eta,z,t) \tag{1}$$

$$\left(a_1\frac{\partial}{\partial t}+a_2\right)T(\xi_0,\eta,z,t)=0 \tag{2}$$

$$T(\xi,\eta,\ell/2,t)=T_0\,e^{-\omega t}f(\xi,\eta) \tag{3}$$

$$T(\xi,\eta,-\ell/2,t)=0 \tag{4}$$

in which $T(\xi,\eta,z,t)$ is the temperature of the plate at point (ξ,η,z) at t time, T_0 is the temperature at time $t=0$ on the circumference of the elliptical plate of radius ξ_0 on the upper face, $\omega>0$ is a constant, λ is the coefficient of thermal conductivity, $\kappa=\lambda/\rho C$ represents thermal diffusivity in which λ being the thermal conductivity of the material, ρ is the density, C is the calorific capacity, assumed to be constant and h is the metric coefficient given by

$$h^2=2/[c^2(\cosh 2\xi-\cos 2\eta)]. \tag{5}$$

$$\nabla^2 = h^2(\partial_{,\xi\xi}+\partial_{,\eta\eta}) \tag{6}$$

2.2 Associated thermal stress problem

The medium is defined by $0\le\xi\le\xi_0$, $0\le\eta\le 2\pi$, $-\ell/2\le z\le\ell/2$, and compiling various boundary conditions in elliptical coordinates are defined to determine the influence of thermal boundary conditions on the thermal stresses. Since we have assumed that the cylinder is sufficiently thin, we can introduce the assumption that the plane, initially normal to the middle or neutral plane $(z=0)$ before bending, remains straight and normal to the middle surface during the deformation, and the length of such elements are not altered. This means that the axial stress negligible compared to the other stress components may be neglected in the stress-strain relations. Thus, for solving quasi-static thermo-elasticity problem by the displacement potential method [11], we assume the potential function $\phi(\xi,\eta,z,t)$ such that it satisfies the equation given below

$$h^2(\phi_{,\xi\xi}+\phi_{,\eta\eta})=\frac{1+\upsilon}{1-\upsilon}\alpha_t T \tag{7}$$

where υ denotes the Poisson's ratio, α_t the coefficient of linear expansion.

The component of the stresses are represented by the use of the Goodier's potential stress function $\phi(\xi,\eta,z,t)$ are represented as

$$\left.\begin{aligned}(1/h^4)\bar{\sigma}_{\xi\xi} &= -2G(c^2/2)[(\cosh 2\xi - \cos 2\eta)\phi,_{\eta\eta} + \sinh 2\xi\ \phi,_{\xi} - \sin 2\eta\ \phi,_{\eta}],\\ (1/h^4)\bar{\sigma}_{\eta\eta} &= -2G(c^2/2)[(\cosh 2\xi - \cos 2\eta)\phi,_{\xi\xi} - \sinh 2\xi\ \phi,_{\xi} + \sin 2\eta\ \phi,_{\eta}],\\ (1/h^4)\bar{\sigma}_{\xi\eta} &= -2G(c^2/2)[-(\cosh 2\xi - \cos 2\eta)\phi,_{\xi\eta} + \sin 2\xi\ \phi,_{\eta} + \sinh 2\eta\ \phi,_{\xi}]\end{aligned}\right\}\quad (8)$$

It is observed that the displacements and stresses obtained from equation (7) and (8) do not satisfy the boundary conditions i.e. plate should be stress-free. To complement the solution, we find the complementary stresses $\bar{\bar{\sigma}}_{ij}$ satisfying the following relations

$$\bar{\sigma}_{\xi\xi} + \bar{\bar{\sigma}}_{\xi\xi} = 0, \quad \bar{\sigma}_{\xi\eta} + \bar{\bar{\sigma}}_{\xi\eta} = 0 \quad \text{on } \xi = a \tag{9}$$

To solve the isothermal elasticity problem, let us make use of the Airy potential stress function $\chi(\xi, \eta, z, t)$ which satisfies the bilaplacian equation as

$$[h^2(\chi,_{\xi\xi} + \chi,_{\eta\eta})]^2 = 0 \tag{10}$$

Then the complementary stresses in terms of Airy stress function are given by

$$\left.\begin{aligned}(1/h^4)\bar{\bar{\sigma}}_{\xi\xi} &= (c^2/2)[(\cosh 2\xi - \cos 2\eta)\chi,_{\eta\eta} + \sinh 2\xi\ \chi,_{\xi} - \sin 2\eta\ \chi,_{\eta}],\\ (1/h^4)\bar{\bar{\sigma}}_{\eta\eta} &= (c^2/2)[(\cosh 2\xi - \cos 2\eta)\chi,_{\xi\xi} - \sinh 2\xi\ \chi,_{\xi} + \sin 2\eta\ \chi,_{\eta}],\\ (1/h^4)\bar{\bar{\sigma}}_{\xi\eta} &= (c^2/2)[-(\cosh 2\xi - \cos 2\eta)\chi,_{\xi\eta} + \sin 2\xi\ \chi,_{\eta} + \sinh 2\eta\ \chi,_{\xi}]\end{aligned}\right\}\quad (11)$$

Thus, the final stresses can be represented as

$$\left.\begin{aligned}\sigma_{\xi\xi} &= \bar{\sigma}_{\xi\xi} + \bar{\bar{\sigma}}_{\xi\xi},\\ \sigma_{\eta\eta} &= \bar{\sigma}_{\eta\eta} + \bar{\bar{\sigma}}_{\eta\eta},\\ \sigma_{\xi\eta} &= \bar{\sigma}_{\xi\eta} + \bar{\bar{\sigma}}_{\xi\eta}\end{aligned}\right\}\quad (12)$$

The equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

The main results obtained in this paper is published in
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A copy of full-page paper is attached. *ANNEX*

Third Research Problem

Quasi-static transient thermal stresses in a thick annular plate subjected to sectional heat supply

✓ **Abstract**

The principal aim of this paper is to investigate the thermoelastic problems in a thick annular plate subjected to sectional heat supply on the upper surfaces whereas the fixed circular edges are at zero temperature. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

Keywords: thick annular plate, thermo-elasticity, integral transform

2000 Mathematics Subject Classification: 74J25, 74H99

✓ **Introduction**

As a result of the increased usage of industrial and construction materials the interest in the thermal stress problems has grown considerably, typified by the annular fins of heat exchangers and brake disc rotors, because of its elementary geometry. Therefore, a number of theoretical studies concerning them have been reported so far. For example,

Nowacki [6] has determined steady-state thermal stresses in circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge. The direct thermoelastic problem in an annular fin is studied by Wu [10] investigates the transient thermal stresses in an annular fin with its base subjected to a heat flux of a decayed exponential function of time. Wankhede [11] has determined the quasi-static thermal stresses in thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Gogulwar and Deshmukh [3] solved the inverse problem of thermal stresses in a thin annular disc, which was further generalized [2] in direct problem. Chiu and Chen [1] investigated stress-field in an annular fin of temperature-dependent conductivity under a periodic heat transfer boundary condition is analyzed by the Adomian's decomposition method. Recently Ootao et al. [8] performed analysis of a three-dimensional transient thermal stress problem is developed for a nonhomogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and /or outer surfaces. In this paper our attempt has been made to discuss quasi static transient thermal stresses in a thick annular plate $a \le r \le b$

$,-h \le z \le h$ and the result illustrated numerically and graphically by using integral transform technique. No one previously studied such type of problem. This is a new contribution to the field.

- **Formulation of the problem**

Consider a thick annular plate of thickness $2h$, occupying a space D defined by $a \le r \le b,$ $-h \le z \le h$. Let the plate be subjected to a transient asymmetric temperature field on the axial direction & axisymmetric temperature field on the radial direction of the cylindrical coordinate system. Initially the plate is kept at zero temperature the arbitrary heat flux $Qf(r)/\lambda$ is prescribed over the upper surface ($z = h$) and the lower surface ($z = -h$) the fixed circular edge ($r = a$ and $r = b$) are at zero temperature. Assume the upper and lower surface of thick annular plate are traction–free surface under this realistic prescribed condition the quasi-static transient thermal stresses are required to be determined.

- **Temperature distribution**

The transient heat conduction equation is given as follows

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

in which thermal diffusivity of the material of the plate is denoted as $\kappa = \lambda / \rho C,$ λ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant, subjected to the initial and boundary conditions as

$$T = 0 \qquad \text{at } \; t = 0 \tag{2}$$

$$T = 0 \qquad \text{at } \; r = a, \; -h \le z \le h \;, \; t > 0 \tag{3}$$

$$T = 0 \qquad \text{at } \; r = b, \; -h \le z \le h \;, \; t > 0 \tag{4}$$

$$T = 0 \qquad \text{at } \; z = -h, \quad a \le r \le b \;, \; t > 0 \tag{5}$$

$$\frac{\partial T}{\partial z} = (Q/\lambda) f(r) t, \qquad \text{at } \; z = h, \quad a \le r \le b \;, \; t > 0 \tag{6}$$

- **Thermal displacements and thermal stress**

The Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [5]

$$\begin{aligned} &\nabla^2 u_r - \frac{u_r}{r} + \frac{1}{1-2\upsilon}\frac{\partial e}{\partial r} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial r} = 0 \\ &\nabla^2 u_z - \frac{1}{1-2\upsilon}\frac{\partial e}{\partial z} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial z} = 0 \end{aligned} \tag{7}$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential ϕ and Love's function L as [4]

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z},$$
$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial^2 z} \tag{8}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t T \tag{9}$$

and the Love's function L must satisfy the equation

$$\nabla^2(\nabla^2 L) = 0 \tag{10}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The component of the stresses are represented by the use of the potential ϕ and Love's function L as

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\upsilon \nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\},$$
$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\upsilon \nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right)\right\},$$
$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\},$$
$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \tag{11}$$

in which G and υ are the shear modulus and Poisson's ratio respectively.

The boundary condition on the traction free surface stress functions are

$$\sigma_{rr} = \sigma_{rz} = 0 \qquad \text{at } z = \pm h \tag{12}$$

Equations (1) to (16) constitute the mathematical formulation of the problem.

The main results obtained in this paper is published in

Annals of Faculty Engineering Hunedoara –International Journal of Engineering Tome XIV (2016) –FASCICULE 3 (AUGUST) ISSN 1584-2665

3) Has the progress been according to original plan of work towards achieving the objects, if not, state reasons.

Yes, the work done so far is according to original plan of work i.e.

To investigate theoretically the thermoelastic behaviour of a radially polarized functionally graded cylindrical vessel subjected to the thermal shock of a transitory temperature change produced by a sudden electric current pulse or radiant energy.

4) Please indicate the difficulties, if any, experienced in implementing project

Due to short funding unable to purchase legal Software such as MATHCAD, MiKTeX, MAPPLE also some useful international journal and proceedings.

5) If project has not been completed, please indicate the approximate time it is likely to be completed. A summary of the work done for the period (Annual basis) may please be send to the commission on a separate sheet.

Not Applicable

6) If the project has been completed, A summary of the finding of the study. Two bound copies of the final report of work done may also be send to the commission.

Two bound copies of summary of final report attached.

7) Any other information, which would help in evaluation of work done on the project. At the completion of the project, the first report should indicate the output, such as (a) Manpower trained. (b) Ph.D. awarded. (c) Publication of the result. (d) other impact, if any.

While working in project work, I learn MATHCAD, MiKTeX, MAPPLE etc software and trained in computing graphs, mathematical evolutions.
I am awarded with Ph.D. which is extention work of project
Also, the results obtained in project work are published in reputed journals and presented in international conference.

Signature of Principal investigator
Dr. Ishaque A. Khan

Signature of Principal
Dr. L. H. Khalsa

BRIEF OBJECTIVE OF THE PROJECT

✓ FOLLOWING ARE THE OBJECTIVES FOR THE ENTIRE TENURE: -

1. To undertake detailed examination of thermoelastic problems several isotropic bodies.
2. To construct the mathematical models on various solids and to discuss the thermoelastic behavior.
3. To study of existing methods and to develop the new technique to find the solution of some thermoelastic problems.
4. Numerical and computational programming is needed while studying the thermoelastic problems.
5. Applications are to be discussed.

✓ WHETHER OBJECTIVES WERE ACHIEVED

Yes…. (Copy Attached) (Annexure)

The work done is according to original plan of work i.e. As per the objective of minor research project we consider the 03 problems on Thermo-elasticity Deformation and examine the results numerically by considering the special cases.

✓ ACHIEVEMENTS FROM THE PROJECT

(Copy Attached) (Annexure)

Some operating possible uses and applications of the present work listed below.

1. This study will be an excellent contribution to the literature in the field of Aeronautical engineering, Production engineering, Structural engineering, etc.
2. The contribution in this form of this work will have its own impact in fields and allied areas, both fundamental and applied and may give a further platform for further research.
3. The study of vibration in thin plate structures in aircraft structure will be subsidiary.
4. The work will open incipient platform and vision of cerebrations on thermoelastic piezo-electro, magneto-electro, arbitrary recollection variable, fraction thermo-elasticity for further research work.
5. This model will be used to predict the behaviour of solid materials under different kind of thermal loading environment.
6. The study will find applications in civil, manufacturing engineering, packaging engineering, material science and in micro-electronics.
7. The study will find applications in processes where mechanical responses of a body is strongly dependent on temperature variations such as post-solidification, buckling, cooling of welds etc.
8. This study will be useful for research in material science, design of new materials, low temperature physics as well as in the advancement of the theory of thermo-elasticity.
9. Mathematical view will enhance material modeling at different scales presents new challenges in developing more sophisticated and accurate computational techniques.
10. Thermoelastic problems of thick and thin plate with heat source with a mathematical view so as to be beneficial in computer programming and industrial applications and develop other approaches for such problem particularly boundary value problems, which provide invaluable check on the accuracy of numerical or approximate schemes and allow for widely applicable parametric studies.

SUMMARY OF THE FINDINGS

Problem 1

Inverse quasi–static unsteady–state thermal stresses in a thick circular plate.

In this problem, a thick circular plate is considered which is kept traction free as well as subjected to arbitrary known interior temperature and determined the expressions for unknown temperature, displacements and stress functions, due to the unknown temperature. As a special case, mathematical model is constructed for $f(r) = (r^2 - a^2)^2 (1 - e^t)$ and numerical calculations performed. The thermoelastic behavior such as temperature, displacements and stresses is examined with the help of arbitrary known interior temperature along the radial direction as $a \to -0.2, b \to -0.1, c \to 0, d \to 0.1$.

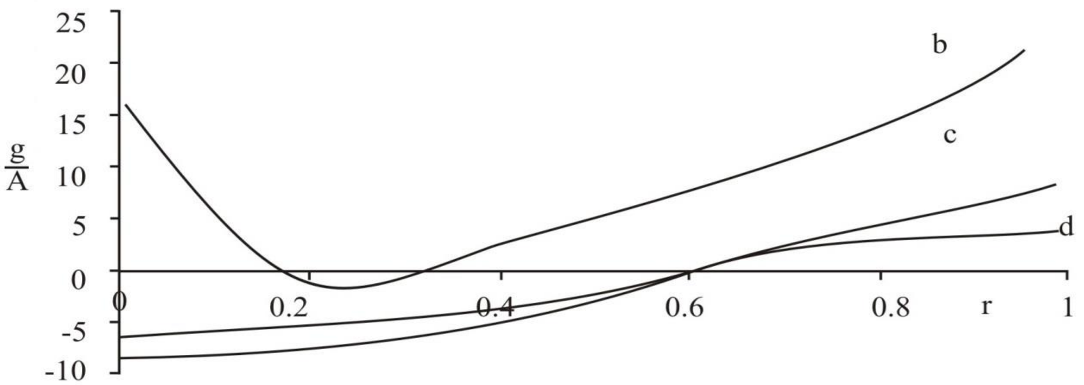

Figure 2. Temperature distribution

Figure 2 indicates that the unknown temperature decreases from r = 0 to r = 0.3 and increases from 0.3 to 1 with the thickness of the circular plate. As the source of known temperature varies from negative to positive value, the unknown temperature decreases its magnitude along radial direction.

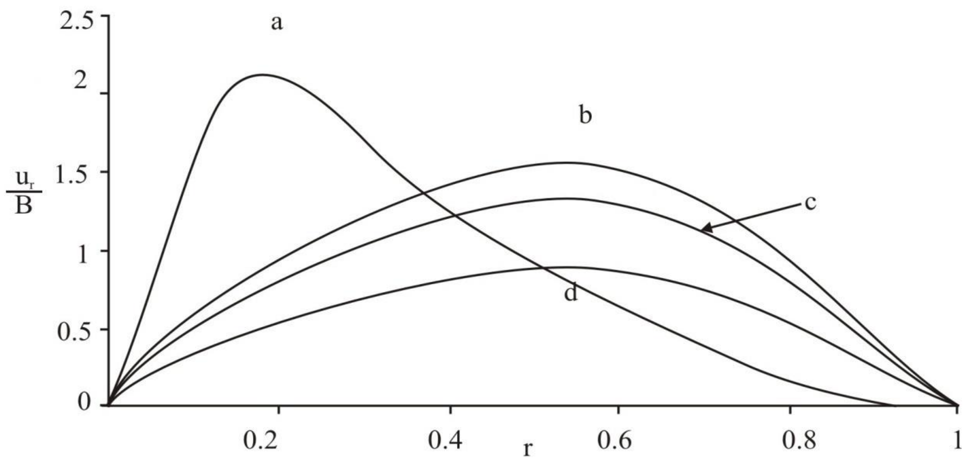

Figure 3. Radial displacement profile

As shown in Figure 3, the source of known temperature varies from bottom to top, the radial displacement decreases at r = 0 and the radial displacement vanishes, or else, its existence would have been visible.

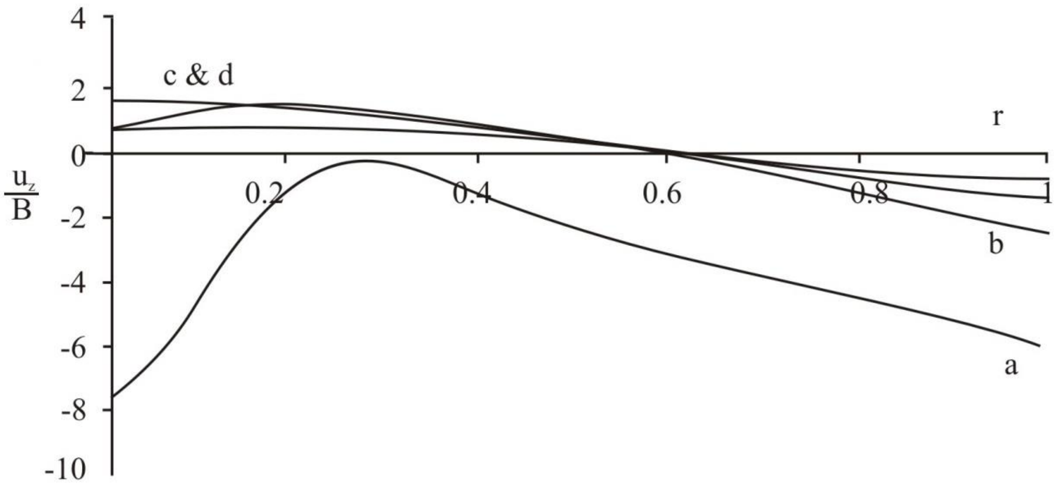

Figure 4. Axial displacement profile

As shown in Figure 4, the source of known temperature varies from bottom to top; the axial displacement increases along radial direction and it shows its existence.

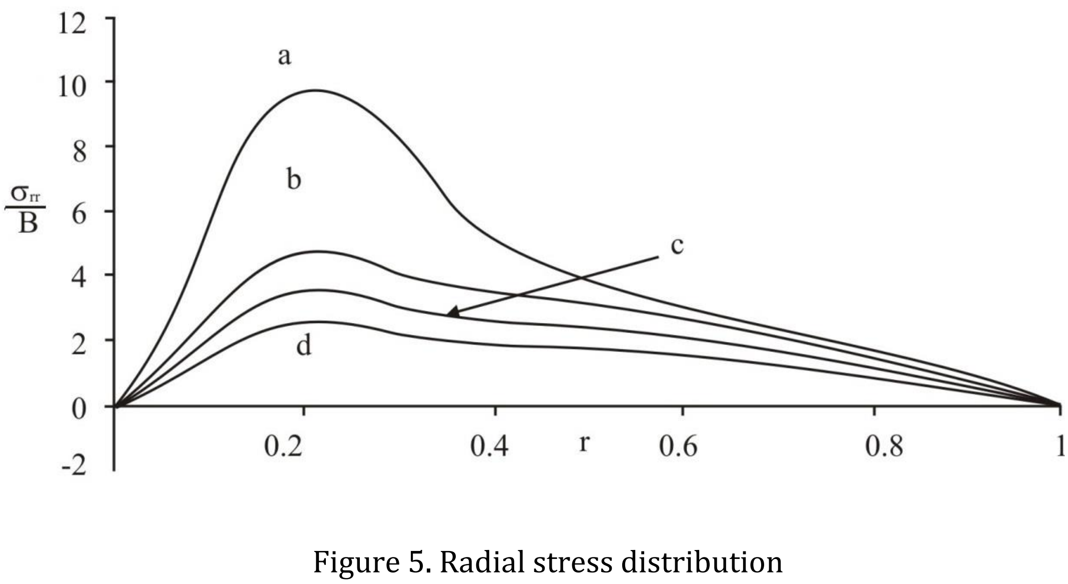

Figure 5. Radial stress distribution

Figure 5 shows that the radial stress decreases from bottom to (lower surface to upper surface) Stress at r = 0 and r = a is zero, otherwise it shows its existence.

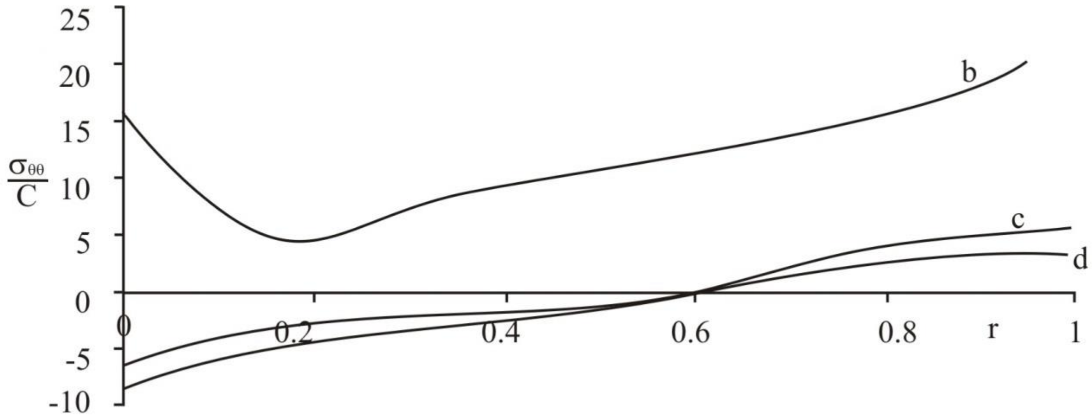

Figure 6. Tangential stress distribution

Figure 6 indicates that the stress function $\sigma_{\theta\theta}$ decrease with the thickness of the circular plate. It shows the existence for small thickness. Also, it develops the tensile stresses in radial direction.

In this article, we analyzed an inverse thermoelastic problem of a thick circular plate and determined the expressions of unknown temperature, displacement, and thermal stresses. The heat conduction differential equation is solved by using finite Hankel and Laplace integral transform techniques, and their inversion theorems. Goodier's and Michell's functions are used to obtain the displacement components. As a special case, a mathematical model is constructed for steel (SN 50C) thick plate, with the material properties specified as above and examined the thermoelastic behaviors in unsteady-state field for unknown temperature change, displacement, and thermal stresses. We conclude that, the displacement and stress components occur near heat source region. With the temperature increase, the circular plate will tend to expand radial direction as well as in axial direction. Also, any particular case of special interest can be derived by assigning values to the parameters and functions in the expressions (19-28).

<u>These Findings are published in-</u>

Applied & Interdisciplinary Mathematics Cogent Mathematics

(2017), 4: 1283763

Taylor & Francis ISSN 2574-2558

http://dx.doi.org/10.1080/23311835.2017.1283763

Problem 2

Quasi-static transient thermal stresses in an elliptical plate due to sectional heat supply on the curved surfaces over the upper face.

In this article, we have described the theoretical treatment of quasi-static thermal stresses in a thin elliptical plate. The temperature distribution and the stresses in the form of ordinary and modified Mathieu functions are used to determine the solution by classical methods. The analytical technique proposed here is relatively straightforward and widely applicable compared to the methods proposed by other researchers. Under given results were obtained while carrying research that can be generalized as follows,

• The advantage of this approach is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions during induced stresses under thermal loading.
• The maximum tensile stress shifting from the outer surface due to maximum expansion of an outer part of the plate, its absolute value increases with the radius. This shifting of stress could be due to heat, stress, concentration or available internal heat sources under known temperature field.
• Finally, the maximum tensile stress occurs in the circular core on the major axis as compared to the elliptical core indicating the distribution of weak heating. This difference might be due to insufficient penetration of heat through the elliptical inner surface.
• The aforementioned thermal stress calculation concept can be very useful in the field of micro-devices or microsystem applications, planar continuum robots, predicting the elastoplastic bending and so forth.

<u>These findings are Published in</u>

Journal of Applied and Computational Mechanics.

4(1) (2018) Page No. 27-39

DOI: 10.22055/JACM.2017.22068.1123 ISSN: 2383-4536

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Problem 3

Quasi-static transient thermal stresses in a thick annular plate subjected to sectional heat supply.

In this study, we have treated thermoelastic problem of a thick annular plate which is considered traction free. We successfully established and obtained the expressions for temperature distribution, displacement and stress function due asymmetric arbitrary heat flux. Then, in order to examine the validity of boundary value problem, we analyze, as a particular case with mathematical model for $f(r) = (r^2 - a^2)(r^2 - b^2)$ and numerical calculations were carried out. The thermoelastic behavior is examined such as temperature, displacement and stresses with the help of arbitrary heat flux at upper surface applied.

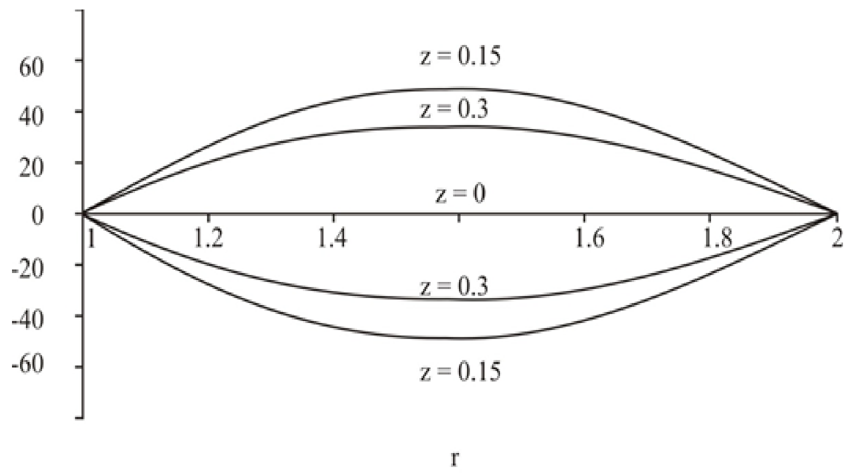

Figure 1: Axial displacement profile along axial direction

Fig. 1 shows the axial displacement u_z occurs at the center i.e. r = 1.5 in radial direction where as in radial direction decreases from lower surface to upper surface.

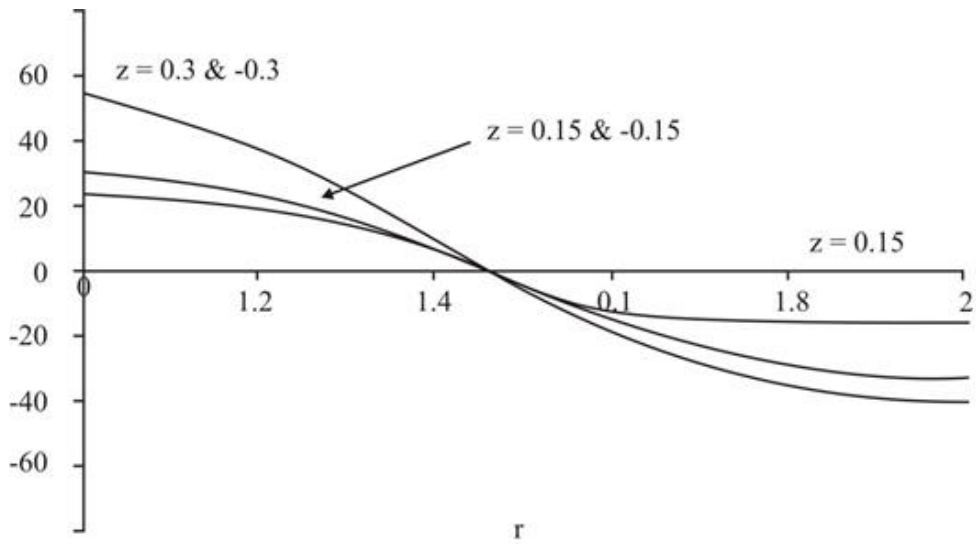

Figure 2: Radial displacement profile along axial direction

As shown in Fig. 2 the variation of thermal stress in the radial displacement u_r decreases from inner circular surface to outer circular surface in radial direction where as in axial direction it take place at upper and lower surfaces of the plate.

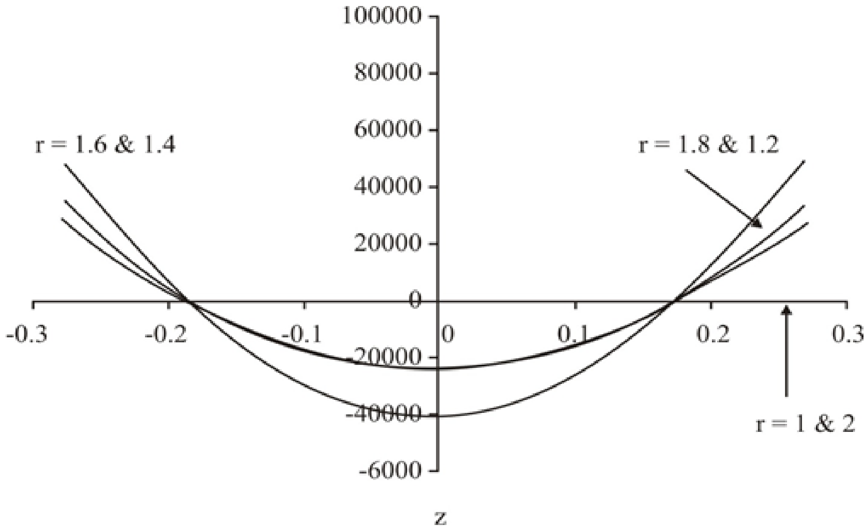

Figure 3: Radial stress distribution along radial direction

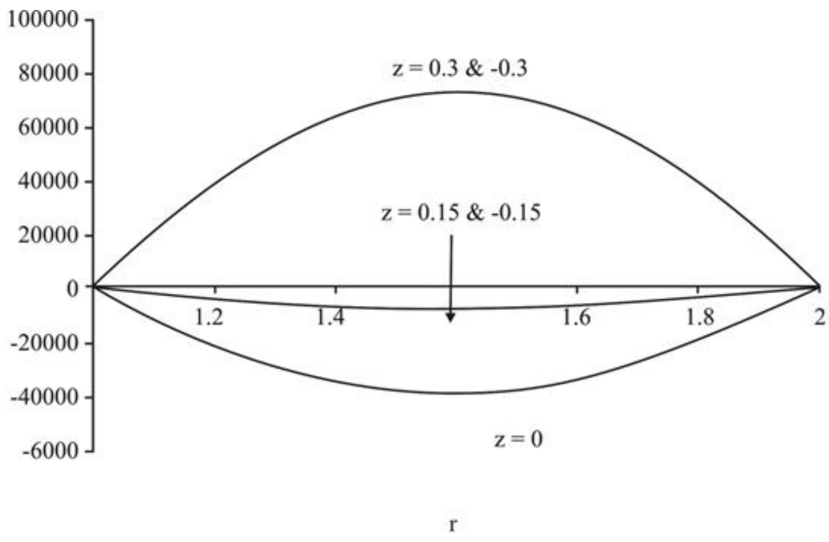

Figure 4: Radial stress distribution along axial direction

Fig. 3 and 4 shows the radial stress function σ_{rr} develops tensile stress at upper and lower surface of the plate, where as it develops compressive stress in the middle of plate.

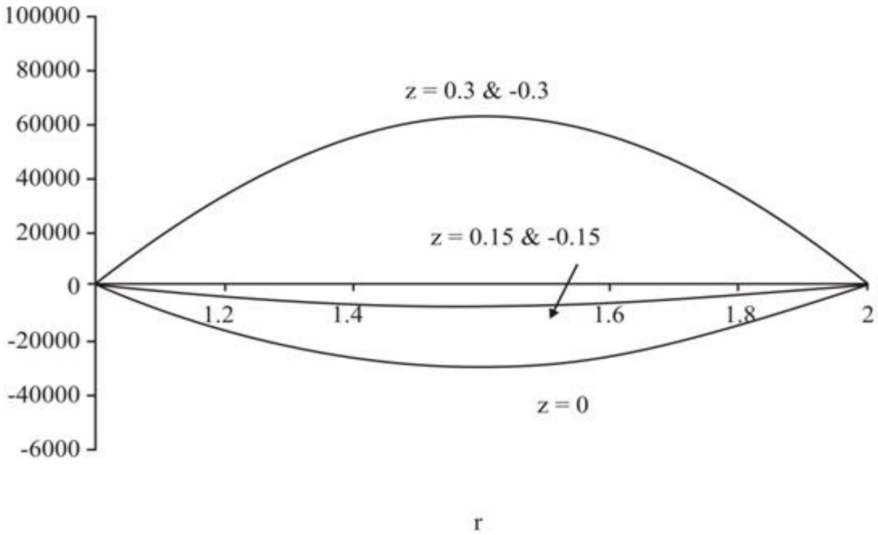

Figure 5: Tangential stress distribution along radial direction

Fig. 5 shows the variation of the stress function $\sigma_{\theta\theta}$ develops tensile stress at the upper and lower surface of the plate where as it develops compressive stress in the middle of plate. We may conclude that the system of equations proposed in this study can be adapted to design of useful structures or machines in engineering applications in the determination of thermoelastic behaviour at every instant and at all points of thick annular disc of finite height.

These findings are published in-

Annals of Faculty Engineering Hunedoara –International Journal of Engineering Tome XIV(2016) –Fascicule 3 (August) ISSN 1584-2665

Annexure

CONTRIBUTION TO THE SOCIETY

- Mathematical modeling continues to be a major component of computer aided engineering and manufacturing.
- A good understanding of (a) the mathematics (b) the physics and (c) the integral method being used to analyze the process is essential for an accurate and efficient solution. Thus, engineers with good background in particular engineering subjects as well as in computational mechanics will continue to have excellent opportunities.
- Problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure.